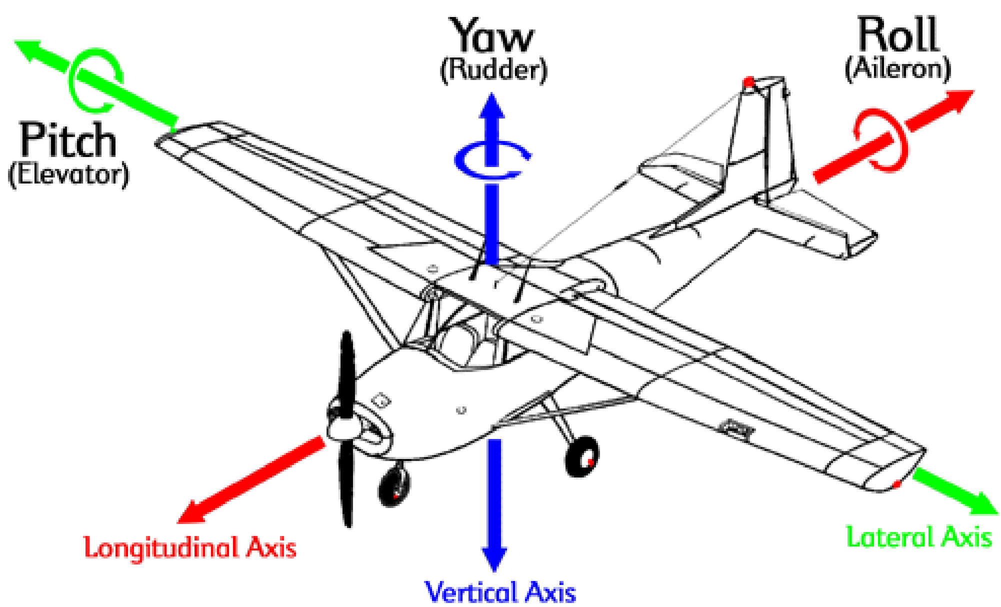
Lecture AC 2

Aircraft Longitudinal Dynamics

- Typical aircraft open-loop motions
- Longitudinal modes
- Impact of actuators
- **Linear Algebra in Action!**



Longitudinal Dynamics

- For notational simplicity, let $X = F_x$, $Y = F_y$, and $Z = F_z$

$$X_u \equiv \left(\frac{\partial F_x}{\partial u}\right), \ldots$$

- Longitudinal equations (1–15) can be rewritten as:

$$\begin{aligned} m\dot{u} &= X_u u + X_w w - mg\cos\Theta_0\theta + \Delta X_c \\ m(\dot{w} - qU_0) &= Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q - mg\sin\Theta_0\theta + \Delta Z_c \\ I_{yy}\dot{q} &= M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + \Delta M_c \end{aligned}$$

  – There is no roll/yaw motion, so $q = \dot{\theta}$.

  – The control commands $\Delta X_c \equiv \Delta F_x^c$, $\Delta Z_c \equiv \Delta F_z^c$, and $\Delta M_c \equiv \Delta M^c$ have not yet been specified.

- Rewrite in **state space** form as

$$\begin{bmatrix} m\dot{u} \\ (m - Z_{\dot{w}})\dot{w} \\ -M_{\dot{w}}\dot{w} + I_{yy}\dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -mg\cos\Theta_0 \\ Z_u & Z_w & Z_q + mU_0 & -mg\sin\Theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X_c \\ \Delta Z_c \\ \Delta M_c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m - Z_{\dot{w}} & 0 & 0 \\ 0 & -M_{\dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -mg\cos\Theta_0 \\ Z_u & Z_w & Z_q + mU_0 & -mg\sin\Theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X_c \\ \Delta Z_c \\ \Delta M_c \\ 0 \end{bmatrix}$$

$$\begin{aligned} E\dot{X} &= \hat{A}X + \hat{\mathbf{c}} \qquad \text{descriptor state space form} \\ \dot{X} &= E^{-1}(\hat{A}X + \hat{\mathbf{c}}) = AX + \mathbf{c} \end{aligned}$$

- Write out in state space form:

$$A = \left[\begin{array}{c|c|c|c} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\Theta_0 \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ I_{yy}^{-1}\left[M_u + Z_u\Gamma\right] & I_{yy}^{-1}\left[M_w + Z_w\Gamma\right] & I_{yy}^{-1}\left[M_q + (Z_q + mU_0)\Gamma\right] & -I_{yy}^{-1} mg\sin\Theta\Gamma \\ 0 & 0 & 1 & 0 \end{array}\right]$$

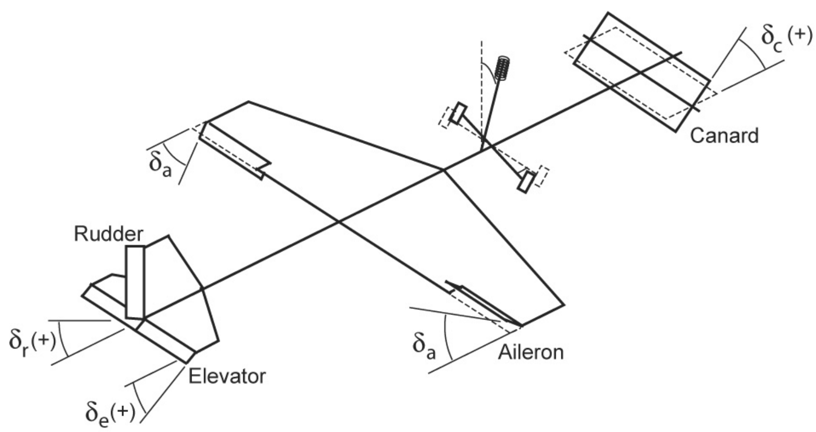
$$\Gamma = \frac{M_{\dot{w}}}{m - Z_{\dot{w}}}$$

- To figure out the **c** vector, we have to say a little more about how the control inputs are applied to the system.

Longitudinal Actuators

- Primary actuators in longitudinal direction are the elevators and the thrust.
  - Clearly the thrusters/elevators play a key role in defining the steady-state/equilibrium flight condition
  - Now interested in determining how they also influence the aircraft motion about this equilibrium condition

$$\text{deflect elevator} \quad \rightarrow \quad u(t), w(t), q(t), \ldots$$



- Recall that we defined ΔX_c as the perturbation in the total force in the X direction as a result of the actuator commands
  - Force change due to an actuator deflection from trim

- Expand these aerodynamic terms using the same perturbation approach

$$\Delta X_c = X_{\delta_e}\delta_e + X_{\delta_p}\delta_p$$

  - δ_e is the deflection of the elevator from trim (down positive)
  - δ_p change in thrust
  - X_{δ_e} and X_{δ_p} are the **control stability derivatives**

- Now we have that

$$\mathbf{c} \;=\; E^{-1}\begin{bmatrix}\Delta X_c\\ \Delta Z_c\\ \Delta M_c\\ 0\end{bmatrix} = E^{-1}\begin{bmatrix}X_{\delta_e} & X_{\delta_p}\\ Z_{\delta_e} & Z_{\delta_p}\\ M_{\delta_e} & M_{\delta_p}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta_e\\ \delta_p\end{bmatrix} = Bu$$

- For the longitudinal case

$$B = \left[\begin{array}{c|c}\frac{X_{\delta_e}}{m} & \frac{X_{\delta_p}}{m}\\ \\ \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} & \frac{Z_{\delta_p}}{m-Z_{\dot{w}}}\\ \\ I_{yy}^{-1}\left[M_{\delta_e}+Z_{\delta_e}\Gamma\right] & I_{yy}^{-1}\left[M_{\delta_p}+Z_{\delta_p}\Gamma\right]\\ \\ 0 & 0\end{array}\right]$$

- Typical values for the B747

$$\begin{array}{ll}X_{\delta_e} = -16.54 & X_{\delta_p} = 0.3mg = 849528\\ Z_{\delta_e} = -1.58\cdot 10^6 & Z_{\delta_p} \approx 0\\ M_{\delta_e} = -5.2\cdot 10^7 & M_{\delta_p} \approx 0\end{array}$$

- Aircraft response $y = G(s)u$

$$\begin{array}{l}\dot{X} = AX + Bu \;\;\rightarrow\;\; G(s) = C(sI-A)^{-1}B\\ y = CX\end{array}$$

- We now have the means to modify the dynamics of the system, but first let's figure out what δ_e and δ_p really do.

Longitudinal Response

- **Final response** to a step input $u = \hat{u}/s$, $y = G(s)u$, use the **FVT**

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\left(G(s)\frac{\hat{u}}{s}\right)$$

$$\Rightarrow \lim_{t\to\infty} y(t) = G(0)\hat{u} = -(CA^{-1}B)\hat{u}$$

- **Initial response** to a step input, use the **IVT**

$$y(0^+) = \lim_{s\to\infty} s\left(G(s)\frac{\hat{u}}{s}\right) = \lim_{s\to\infty} G(s)\hat{u}$$

  – For your system, $G(s) = C(sI - A)^{-1}B + D$, but $D \equiv 0$, so

$$\lim_{s\to\infty} G(s) \to 0$$

  – **Note: there is NO immediate change** in the output of the motion variables in response to an elevator input $\Rightarrow y(0^+) = 0$

- Consider the *rate of change* of these variables $\dot{\mathbf{y}}(\mathbf{0}^+)$

$$\dot{y}(t) = C\dot{X} = CAX + CBu$$

  and normally have that $CB \neq 0$. Repeat process above[1] to show that $\dot{y}(0^+) = CB\hat{u}$, and since $C \equiv I$,

$$\dot{y}(0^+) = B\hat{u}$$

- Looks good. Now compare with numerical values computed in MATLAB®
  Plot u, α, and flight path angle $\gamma = \theta - \alpha$ (since $\Theta_0 = \gamma_0 = 0$)
  See AC 1-10

[1] Note that $C(sI - A)^{-1}B + D = D + \frac{CB}{s} + \frac{CA^{-1}B}{s^2} + \ldots$

Elevator (1° elevator down – stick forward)

- See very rapid response that decays quickly (mostly in the first 10 seconds of the α response)
- Also see a very lightly damped long period response (mostly u, some γ, and very little α). Settles in >600 secs
- Predicted **steady state** values from code:

14.1429	m/s	u	(speeds up)
-0.0185	rad	α	(slight reduction in AOA)
-0.0000	rad/s	q	
-0.0161	rad	θ	
0.0024	rad	γ	

  - Predictions appear to agree well with the numerical results.
  - **Primary result** is a **slightly lower angle of attack and a higher speed**

- Predicted **initial rates** of the output values from code:

-0.0001	m/s^2	$\dot{u}$
-0.0233	rad/s	$\dot{\alpha}$
-1.1569	rad/s^2	$\dot{q}$
0.0000	rad/s	$\dot{\theta}$
0.0233	rad/s	$\dot{\gamma}$

  - All outputs are at zero at $t = 0^+$, but see rapid changes in α and q.
  - Changes in u and γ (also a function of θ) are much more gradual – not as easy to see this aspect of the prediction

- **Initial impact** Change in α and q (pitches aircraft)
- **Long term impact** Change in u (determines speed at new equilibrium condition)

Thrust (1/6 input)

- Motion now dominated by the lightly damped long period response
- Short period motion barely noticeable at beginning.
- Predicted **steady state** values from code:

0	m/s	u
0	rad	α
0	rad/s	q
0.05	rad	θ
0.05	rad	γ

  – Predictions appear to agree well with the simulations.
  – **Primary result** is that we are **now climbing with a flight path angle of 0.05 rad at the same speed we were going before.**

- Predicted **initial rates** of the output values from code:

2.9430	m/s^2	$\dot{u}$
0	rad/s	$\dot{\alpha}$
0	rad/s^2	$\dot{q}$
0	rad/s	$\dot{\theta}$
0	rad/s	$\dot{\gamma}$

  – Changes to α are very small, and γ response initially flat.
  – Increase power, and the aircraft initially speeds up

- **Initial impact** Change in u (accelerates aircraft)
- **Long term impact** Change in γ (determines climb rate)

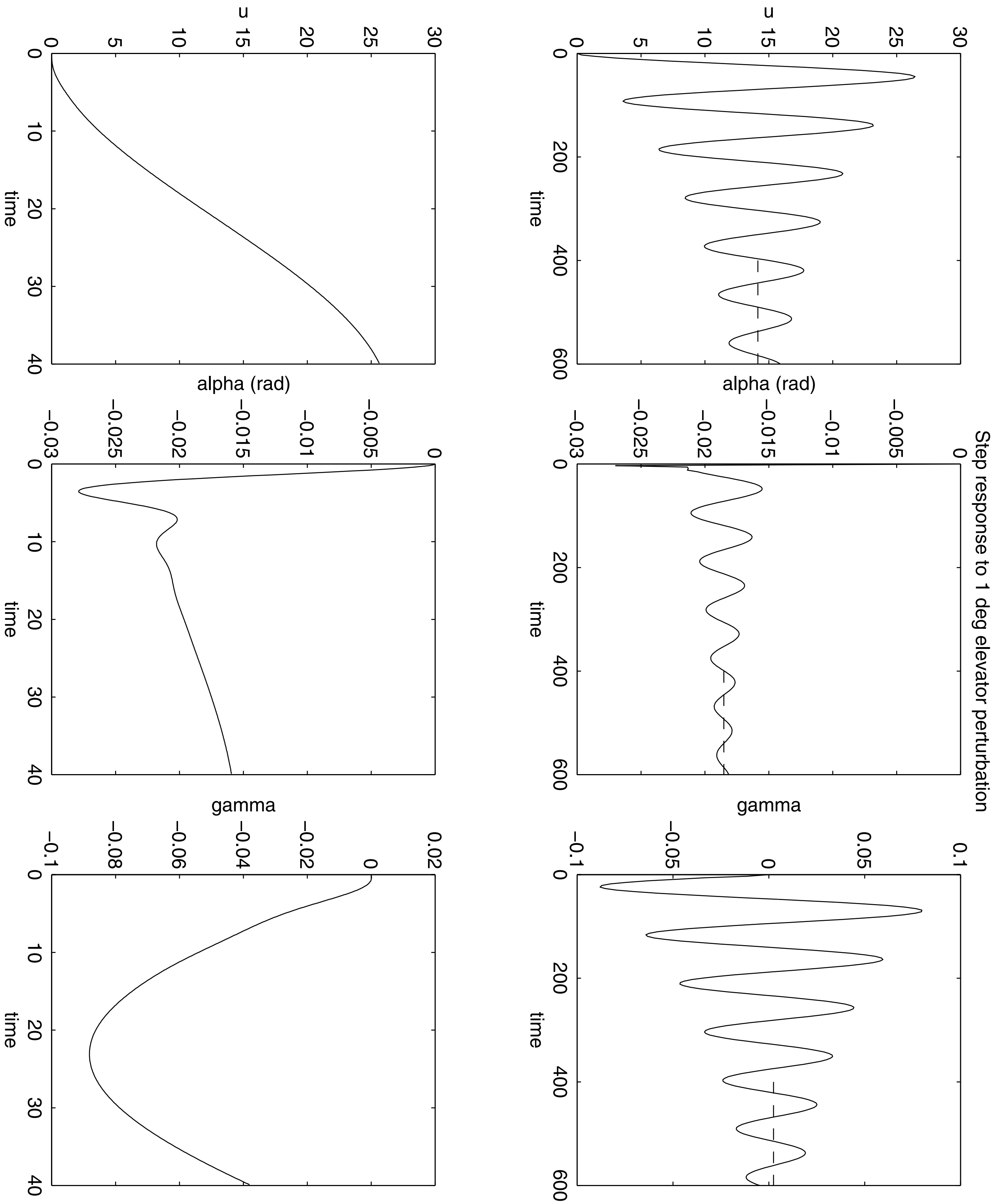


Figure 1: Step Response to 1 deg elevator perturbation – B747 at M=0.8

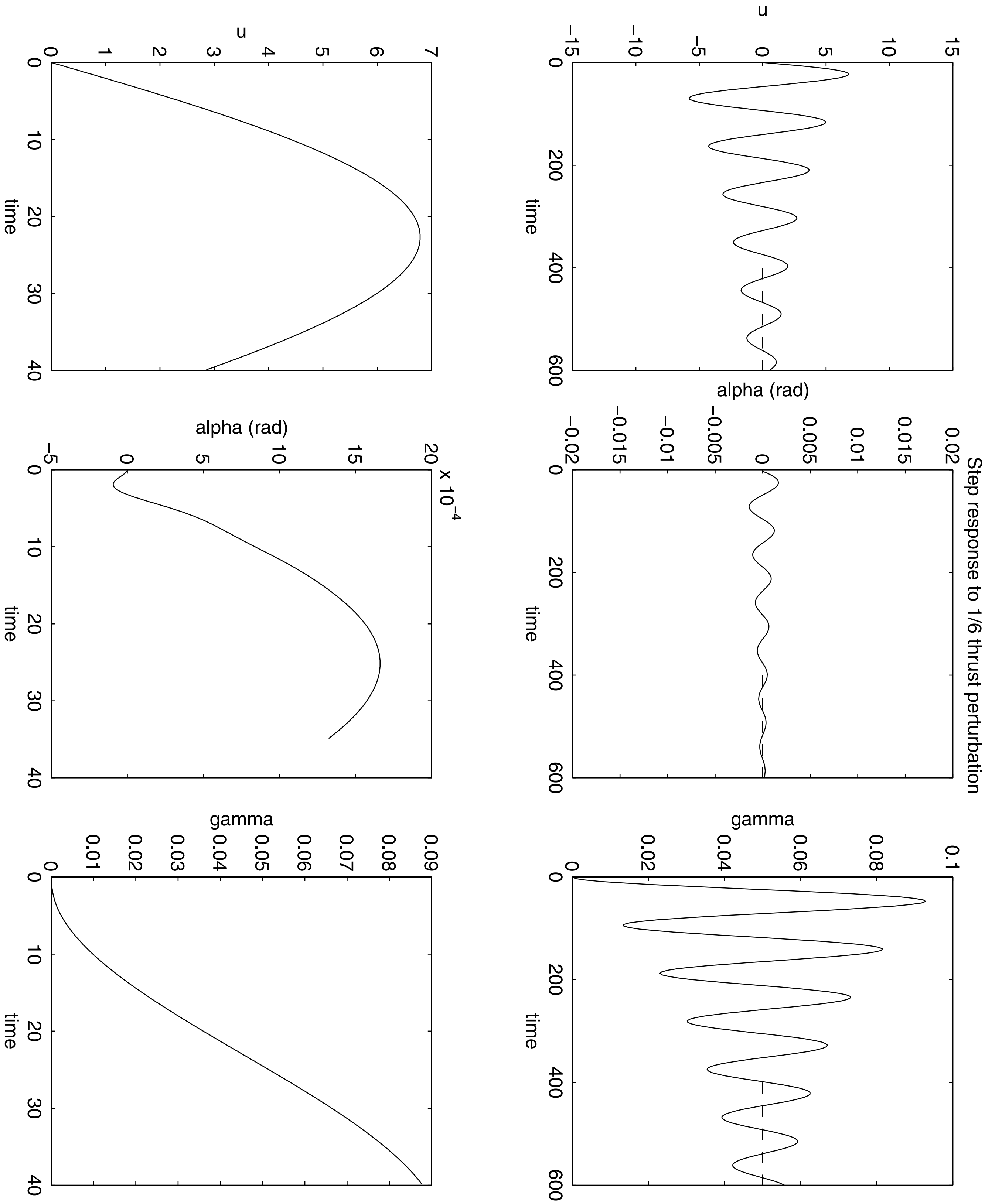


Figure 2: Step Response to 1/6 thrust perturbation – B747 at M=0.8

- **Summary:**

  – **To increase equilibrium climb rate, add power.**

  – **To increase equilibrium speed, increase δ_e (move elevator further down).**

  – Transient (initial) effects are the opposite **and tend to be more consistent with what you would intuitively expect to occur**

Modal Behavior

- Analyze the model of the vehicle dynamics to quantify the responses we saw.
  - Homogeneous dynamics are of the form $\dot{X} = AX$, so the response is
  $$X(t) = e^{At}X(0) \ \text{– a matrix exponential.}$$

- To simplify the investigation of the system response, find the **modes** of the system using the *eigenvalues* and *eigenvectors*
  - λ is an **eigenvalue** of A if
  $$\det(\lambda I - A) = 0$$
  which is true iff there exists a nonzero v (**eigenvector**) for which
  $$(\lambda I - A)v = 0 \quad \Rightarrow \quad Av = \lambda v$$
  - If A $(n \times n)$, typically will get n eigenvalues and eigenvectors $Av_i = \lambda_i v_i$
  - Assuming that the eigenvectors are **linearly independent**, can form
  $$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$
  $$AT = T\Lambda$$
  $$\Rightarrow T^{-1}AT = \Lambda \quad , \quad A = T\Lambda T^{-1}$$
  - Given that $e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots$, and that $A = T\Lambda T^{-1}$, then it is easy to show that
  $$X(t) = e^{At}X(0) = Te^{\Lambda t}T^{-1}X(0) = \sum_{i=1}^{n} v_i e^{\lambda_i t}\beta_i$$
  - **State solution is a linear combination of the system modes $v_i e^{\lambda_i t}$**

$e^{\lambda_i t}$ – determines the **nature** of the time response

v_i – determines the extent to which each state **contributes** to that mode

β_i – determines the extent to which the initial condition **excites** the mode

- Thus the total behavior of the system can be found from the system modes

- Consider numerical example of B747

$$A = \begin{bmatrix} -0.0069 & 0.0139 & 0 & -9.8100 \\ -0.0905 & -0.3149 & 235.8928 & 0 \\ 0.0004 & -0.0034 & -0.4282 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

which gives two sets of complex eigenvalues

$$\lambda = -0.3717 \pm 0.8869\mathbf{i}, \quad \omega = 0.962, \quad \zeta = 0.387, \qquad \text{short period}$$

$$\lambda = -0.0033 \pm 0.0672\mathbf{i}, \quad \omega = 0.067, \quad \zeta = 0.049, \quad \text{Phugoid - long period}$$

  – **result is consistent with step response** - heavily damped fast response, and a lightly damped slow one.

- To understand the eigenvectors, we have to do some normalization (scales each element appropriately so that we can compare relative sizes)
  – $\hat{u} = u/U_0$, $\hat{w} = w/U_0$, $\hat{q} = q/(2U_0/\bar{c})$
  – Then divide through so that $\theta \equiv 1$

	Short Period	Phugoid
$\hat{u}$	$0.0156 + 0.0244\mathbf{i}$	$-0.0254 + 0.6165\mathbf{i}$
$\hat{w}$	$1.0202 + 0.3553\mathbf{i}$	$0.0045 + 0.0356\mathbf{i}$
$\hat{q}$	$-0.0066 + 0.0156\mathbf{i}$	$-0.0001 + 0.0012\mathbf{i}$
θ	1.0000	1.0000

- **Short Period** – primarily θ and $\alpha = \hat{w}$ in the same phase. The $\hat{u}$ and $\hat{q}$ response is very small.

- **Phugoid** – primarily θ and $\hat{u}$, and θ lags by about 90°. The $\hat{w}$ and $\hat{q}$ response is very small $\Rightarrow$ consisitent with approximate solution on AC 2–1?

- Dominant behavior agrees with time step responses – note how initial conditions were formed.

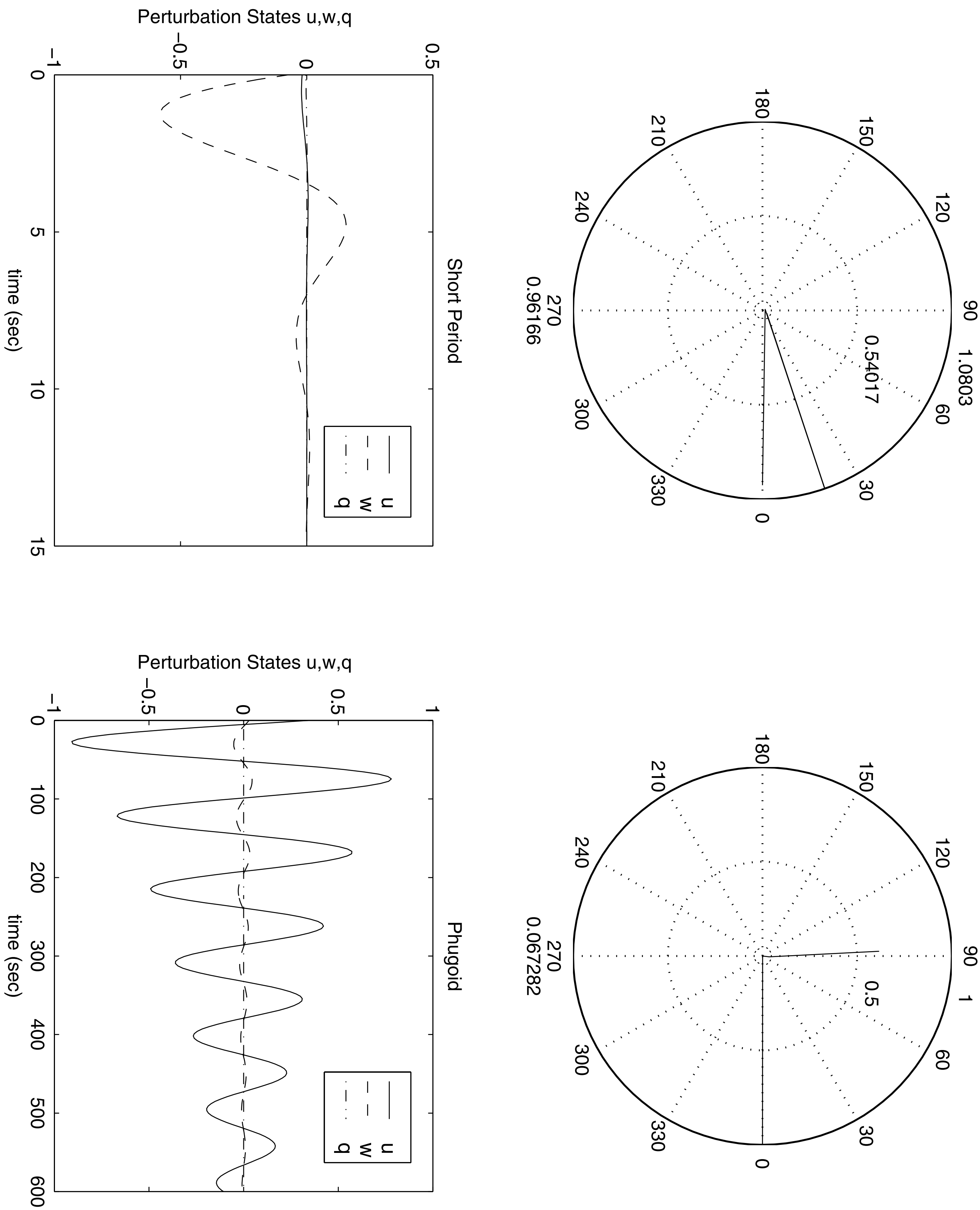


Figure 3: Mode Response – B747 at M=0.8

Summary

- Two primary longitudinal modes: phugoid and short-period
- Impact of the various actuators clarified:
  - Short time-scale
  - Long time-scale