16.512, Rocket Propulsion
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Lecture 7: Convective Heat Transfer: Reynolds Analogy

Heat Transfer in Rocket Nozzles

General

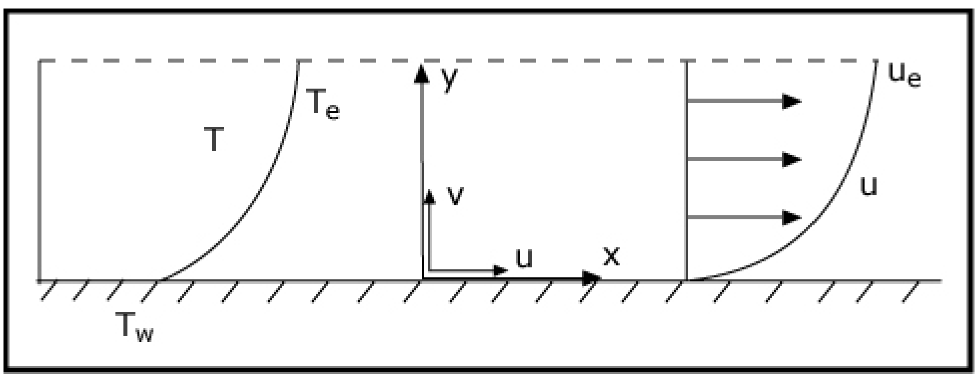
Heat transfer to walls can affect a rocket in at least two ways:

(a) Reducing the performance. This tends to be a 1-3% effect on I_{sp} only, and is therefore secondary.
(b) Creating great difficulties in the design of hot-side structures that have to survive heat fluxes in the $10^7 - 10^8$ w/m^2 range.

The principal modes of heat transfer to nozzle and combustor walls are convection and radiation. Of these, convection dominates, and radiation tends to be important only for particle-laden flows from solid propellant rockets.

Convective Heat Transfer

We will review here the compressible 2D boundary layer equations in order to extract information on wall heat transfer.



The governing equations are (in the B.L. approximation)

Continuity $$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

X-Momentum $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = \frac{\partial \tau_{xy}}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad (2)$$

Y-Momentum $$\frac{\partial P}{\partial y} = 0 \qquad (3)$$

Total enthalpy $$\rho u\frac{\partial h_t}{\partial x}+\rho v\frac{\partial h_t}{\partial y}=\frac{\partial}{\partial y}\left(u\tau_{xy}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \qquad (4)$$

where $h_t = h+\frac{u^2}{2}$ is the specific total enthalpy, and μ is the viscosity. For a laminar flow, $\mu=\mu(T)$ is a fluid property. Rocket boundary layers are almost always turbulent, and μ is then the "turbulent viscosity", where momentum transport is effected by the random motion of turbulent "eddies". If these eddies have a velocity scale u' and a length scale l', we have, in order-of-magnitude.

$$\mu_{turb.}\sim\rho u'l' \qquad (5)$$

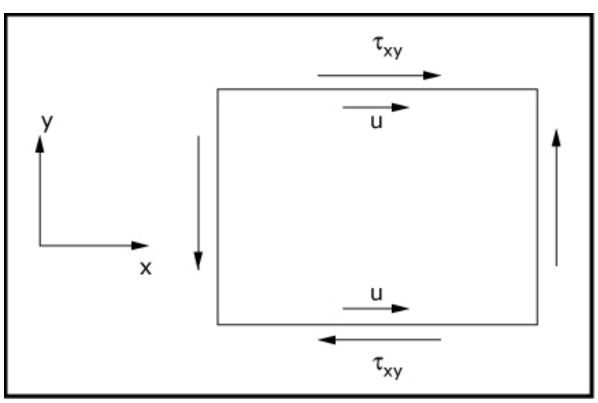
where u' is some fraction of the local u, and l' tends to be of the order of the wall distance y. The important points about (5) are

(a) $\mu_{turb.}\gg\mu$, mostly because $l'\gg$ mean free path and
(b) $\mu_{turb.}$ is proportional to density (whereas μ is not, because the m.f.p. is inversely proportional to ρ).

Similarly, the last term on the right in the energy balance, representing the convergence of heat flux, contains the "turbulent thermal conductivity" $K\sim\rho c_p u'l'$. Once again, we notice that K is here proportional to density. We also note that the "turbulent Prandtl number"

$$P_r=\frac{\mu_t c_p}{k_t}\sim 1 \text{ (from the orders of magnitude)}$$

It is of some interest to note the origin and composition of the viscous term in equation (4). If we collect the dot products $\vec{u}\bullet\vec{\tau}_t$ around a fluid element as shown (in B.L. approximation),



we obtain the term $\frac{\partial}{\partial y}\left(u\tau_{xy}\right)$ as written in (4). This can be expanded as

$$\frac{\partial}{\partial y}\left(u\tau_{xy}\right) = u\frac{\partial\tau_{xy}}{\partial y} + \tau_{xy}\frac{\partial u}{\partial y} = u\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad (6)$$

The 1st term in (6) is just the velocity times the viscous net force per unit volume, so it is the part of the total viscous work that goes to accelerate the local flow. The second term in (6) is positive definite, and it is the rate of dissipation of energy into heat due to viscous effects. We will return later to this heating effect.

Approximate Analysis Let us manipulate the right hand side of equation (4):

$$\frac{\partial}{\partial y}\left(u\mu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left[\mu\left(u\frac{\partial u}{\partial y} + \frac{K}{\mu}\frac{\partial T}{\partial y}\right)\right]$$

and, since $\frac{\partial h}{\partial y} = c_p\frac{\partial T}{\partial y}$, this yields

$$\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u^2/2}{\partial y} + \frac{1}{P_r}\frac{\partial h}{\partial y}\right)\right] \qquad \left(P_r \equiv \frac{\mu c_p}{k}\right)$$

We note here that, both for laminar and turbulent flows, P_r is a constant, independent of P and T to a good approximation. In fact, as we noted before, it is also of order unity (~ 0.9 for turbulent flows). So, the RHS of the energy equation becomes

$$\frac{\partial}{\partial y}\left[\mu\frac{\partial}{\partial y}\left(\frac{h}{P_r} + \frac{u^2}{2}\right)\right] \qquad (7)$$

If we made the approximation $P_r = 1$, then this would reduce further to $\frac{\partial}{\partial y}\left(\mu\frac{\partial h_t}{\partial y}\right)$ with $h_t = h + \frac{u^2}{2}$. If in addition, we made the "flat plate" approximation $\frac{\partial P}{\partial x} \simeq 0$, then the pair of equations (1), (4) would become } approximations

$$\left.\begin{aligned} \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \\ \rho u\frac{\partial h_t}{\partial x} + \rho v\frac{\partial h_t}{\partial y} &= \frac{\partial}{\partial y}\left(\mu\frac{\partial h_t}{\partial y}\right) \end{aligned}\right\} \qquad (8)$$

These are identical equations for u and h_t. The same equation would also govern the linearly transformed variables

$$\tilde{u} = \frac{u}{u_e}; \qquad \tilde{h} = \frac{h_t - h_{t_w}}{h_{t_e} - h_{t_w}} \tag{9}$$

where the $(\;)_e$ subscript denotes the value of a variable in the local "external" flow (just outside the boundary layer). Both $\tilde{u}$ and $\tilde{h}$ satisfy identical boundary conditions:

$$\tilde{u}_w = \tilde{h}_w = 0; \quad \tilde{u}_e = \tilde{h}_e = 1 \tag{10}$$

and, as noted, identical governing equations. We conclude that, under the assumption

$$\left(P_r = 1,\ \frac{\partial P}{\partial x} = 0\right),$$

$$\boxed{\frac{h_t - h_w}{h_{t_e} - h_w} = \frac{u}{u_e}} \tag{11}$$

where we also noticed $h_{tw} = h_w + \cancel{\frac{u_w^2}{2}} = h_w$. This similarity relation between velocity and total enthalpy profiles is known as <u>Crocco's analogy</u>.

<u>Approximate heat flux at the wall</u>

We are interested in the <u>magnitude</u> of the wall heat flux

$$q_w = \left(K\frac{\partial T}{\partial y}\right)_w \tag{12}$$

$$q_w = \left(\frac{K}{c_p}\frac{\partial h}{\partial y}\right)_w = \left(\frac{K}{\mu c_p}\mu\frac{\partial h_t}{\partial y}\right)_w$$

where we used $\left(\frac{\partial h_t}{\partial y}\right)_w = \frac{\partial}{\partial y}\left(h + \frac{u^2}{2}\right)_w = \left(\frac{\partial h}{\partial y}\right)_w + \underbrace{\cancel{\left(u\frac{\partial h}{\partial y}\right)_w}}_{0,\ \text{since } u_w = 0}$

The group $\frac{K}{\mu c_p} = \frac{1}{P_r}$ should be set equal to unity, for consistency with the stated approximations. Thus

$$q_w = \left(\mu \frac{\partial h_t}{\partial y}\right)_w$$

Use now equation (11):

$$q_w = \left(\mu \frac{\partial}{\partial y}\left[h_w + \left(h_{t_e} - h_w\right)\frac{u}{u_e}\right]\right)_w = \frac{h_{t_e} - h_w}{u_e}\left(\mu \frac{\partial u}{\partial y}\right)_w$$

and notice that $\left(\mu \frac{\partial u}{\partial y}\right)_w$ is the wall shear stress, τ_w. So

$$\boxed{q_w = \frac{h_{t_e} - h_w}{u_e}\tau_w} \tag{13}$$

which is also called Reynolds analogy. A more compact form of this can be written in terms of the Friction Coefficient

$$c_f \equiv \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} \tag{14}$$

and the Stanton number

$$S_t = \frac{q_w}{\rho_e u_e \left(h_{t_e} - h_w\right)} \tag{15}$$

with the result (from (13))

$$S_t = \frac{c_f}{2} \tag{16}$$

One important point can be made about the result (13):

The heat flux to the wall is driven by the enthalpy (or temperature) difference between Total external and Wall values, not between static values. This can be non-intuitive. Consider the situation near the exit of a highly expanded space nozzle, where the bulk temperature T_e may have dropped to, say, 300K due to the strong expansion from a chamber temperature of, say, 3000K. The wall could be made of Tungsten so as to be able to sustain relatively high temperature and cool itself by radiation to space, so T_w could be, say, 1500K. Is the nozzle wall being heated or cooled by the 300K gas? The answer is that it is being heated, because

$T_{t_e} = T_c = 3000K$, while $T_w = 1500 < T_{t_e}$. Are we violating the 2nd Principle of Thermodynamics. Read on.

Simplified Profiles, Across the Boundary Layer

To better understand this situation, let us return to Crocco's analogy (equation 11) and write $h_t = h + u^2/2$, and solve for h:

$$h = \left(h_{t_e} - h_w\right)\frac{u}{u_e} - \frac{u^2}{2} \tag{17}$$

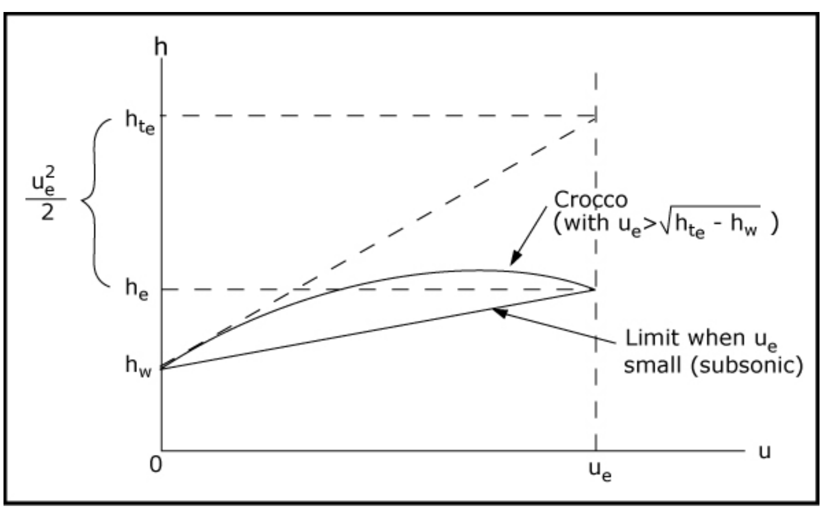
This is a quadratic relationship between h and u. For low subsonic flows $h \simeq h_t$, so the last term is not strong, and the relationship becomes linear in the limit.

The relationship between slopes at the wall flows from (17):

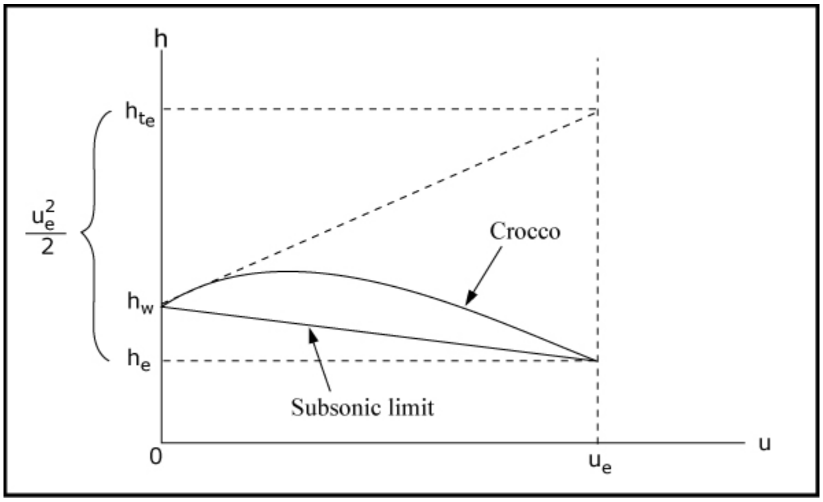
$$\left(\frac{dh}{dy}\right)_w = \left(h_{t_e} - h_w\right)\frac{1}{u_e}\left(\frac{du}{dy}\right)_w - \cancelto{0}{\left(u\frac{\partial u}{\partial y}\right)_w}$$

or
$$\left(\frac{dh}{du}\right)_w = \left(\frac{h_{t_e} - h_w}{u_e}\right) \tag{18}$$

We can use (17) and (18) to sketch h vs. u across the boundary layer. For a case with $h_e > h_w$, this looks like



Note that whenever $u_e > \sqrt{h_{t_e} - h_w}$ there develops an intermediate temperature maximum. But in any case, the wall slope is as if the line were coming from h_{t_e}, not from h_e. The case when $h_e < h_w$ is more revealing even:



Now the wall slope is seen to be positive (heat into the wall), despite $h_e < h_w$ (as long as $h_{t_e} > h_w$)

So, the quadratic portion of the Crocco relationship is responsible for the extra wall heat; this can in turn be traced to viscous dissipation, which accumulates in the boundary layer and elevates its temperature, so that the wall is heated even when the outside temperature is low (as long as the flow has high speed).

Modification for $P_r \neq 1$

We leave for now the issue of the non zero pressure gradient, except to note that it introduces small modifications down to the throat. The deviations of P_r from unity are small, and, for gases $P_r < 1$ (~ 0.9 for turbulent flow). This breaks the perfect balance between dissipation and conduction responsible for Crocco's analogy, in the sense of favoring conduction of the dissipated heat. As a second consequence, the temperature overshoot is reduced, and so is the wall slope of T and the heat flux to the wall. The direct effect of higher conduction ($P_r < 1$) is accounted for approximately by modifying Reynolds analogy to

$$\boxed{S_t = \frac{c_f}{2P_r 0.6}} \tag{19}$$

The secondary effect (reduced overshoot) is accounted for by replacing the driving enthalpy difference $h_{t_e} - h_w$ by $h_{aw} - h_w$, where h_{aw} is the "Adiabatic-wall enthalpy", defined as

$$\boxed{h_{aw} = h_e + r\frac{u_e^2}{2}} \quad ; \; r \simeq 0.9 \tag{20}$$

(turbulent)

and r is the "Recovery factor".

With these changes, the heat flux is now

$$q_w = \rho_e u_e \left(h_{aw} - h_w\right)\frac{c_f}{2P_r 0.6} \tag{21}$$

<u>The Bartz heat flux formula</u>

A very crude, but surprisingly effective representation for the friction factor c_f is that supplied by the well-studied case of fully developed turbulent flow in a pipe.

$$c_f = \frac{0.046}{R_e^{0.2}}; \qquad R_e = \frac{\rho_e u_e D}{\mu_e} \tag{22}$$

where R_e is the Reynolds number based on diameter D, and μ_e is the laminar viscosity. Putting also $h = c_p T +$ constant, equation (21) now gives

$$q_w = \rho_e u_e c_p \left(T_{aw} - T_w\right)\underbrace{\frac{0.023}{P_r 0.6}}_{\simeq 0.026}\left(\frac{\mu_e}{\rho_e u_e D}\right)^{0.2} = \frac{0.026}{D^{0.2}}\left(\rho_e u_e\right)^{0.8}\mu_e^{0.2} c_p\left(T_{aw} - T_w\right) \tag{23}$$

It is common practice to define a heat transfer "gas-side film coefficient", h_g (not an enthalpy!) by

$$h_g \equiv \frac{q_w}{T_{aw} - T_w} \tag{24}$$

And, so far, we have

$$h_g = \frac{0.026}{D^{0.2}}\left(\rho_e u_e\right)^{0.8}\mu_e^{0.2} c_p \tag{25}$$

At this point we note that the formulation so far has ignored the strong variations of ρ and μ across the boundary layer since these quantities depend on temperature as

$$\rho \sim \frac{1}{T} \text{ (at P=constant) ; } \mu \sim T^w \; \left(w \simeq 0.6\right) \tag{26}$$

A commonly used approach to including these variations is to replace ρ_e and μ_e in equation (25) by their values at some intermediate temperature <T>:

$$\rho_e \to \rho_e \frac{T_e}{<T>}; \qquad \mu_e \to \mu_e \left(\frac{<T>}{T_e}\right)^w \tag{27}$$

and <T> can be evaluated by several empirical rules. For Mach numbers not much higher than 1, we can simply use

$$<T> \simeq \frac{T_e + T_w}{2} \tag{28}$$

Making the replacements of (27) in equation (25), we obtain

$$h_g = \frac{0.026}{D^{0.2}} \left(\rho_e u_e\right)^{0.8} \left(\frac{T_e}{<T>}\right)^{0.8-0.2w} \mu_e^{0.2} c_p \tag{29}$$

which is one form of Bartz' formula. A more useful form follows from the continuity equation:

$$\rho_e u_e = \frac{\dot{m}}{A} = \frac{P_c}{c*} \frac{A_t}{A}, \text{ with } \frac{\sqrt{R_g T_c}}{\Gamma(\gamma)},$$

and where A is the local cross-section, and A_t the throat cross-section. Substituting in (29), and using $\frac{A_t}{A} = \left(\frac{D_t}{D}\right)^2$, the final form is

$$\boxed{h_g = \frac{0.026}{D_t^{0.2}} \left(\frac{P_c}{c*}\right)^{0.8} \left(\frac{D_t}{D}\right)^{1.8} c_p \mu_e^{0.2} \left(\frac{T_e}{<T>}\right)^{0.8-0.2w}} \tag{30}$$

Several important trends and observations can be made now:

(a) Smaller throat diameter leads to larger heat flux $\left(\sim \frac{1}{D_t^{0.2}}\right)$. This comes straight from the Reynolds no. dependence of c_f.

(b) Heat flux is <u>almost</u> linear in chamber pressure $\left(\sim P_c^{0.8}\right)$. This limits the feasibility of high chamber pressures, which are otherwise very desirable.

(c) Maximum heat flux occurs at the throat $\left(\sim \left(\frac{D_t}{D}\right)^{1.8}\right)$. One critical design consideration is therefore the thermal integrity of the throat structure.

(d) Lighter gasses lead to higher heat fluxes, through the combined effects of c_p and $c*$ $\left(h_g \sim \frac{1}{M^{0.6}}\right)$

(e) The factor $\left(\frac{T_e}{<T>}\right)^{0.8-0.2w} \simeq \left(\frac{T_e}{<T>}\right)^{0.68}$ is greater than unity. This enhancement of heat flux follows mainly from the fact that the gas in the boundary layer is mostly cooler than in the core, hence denser. We showed before that the turbulent heat conductivity is proportional to density.

Example

Consider the Space Shuttle Main Engine (SSME), which is a Hydrogen-Oxygen rocket with (roughly) these characteristics:

$$P_c = 220\,\text{atm} \simeq 2.2 \times 10^7 P_a$$

$$T_c = 3600\,K$$

$$M = 15g/mol$$

$$r \simeq 1.25$$

$$c^* = \frac{\sqrt{R_g T_c}}{\Gamma(\gamma)} \simeq 2600\,m/s$$

$$c_p = \frac{\gamma}{\gamma - 1}\frac{R}{M} \simeq 2800\ J/Kg/K$$

$$\mu_e \simeq 3 \times 10^{-5}\,Kg/m/s$$

$$T_{throat} = \frac{2}{\gamma + 1} T_c \simeq 3200\,K = T_e$$

$$T_w = 1000\,K$$

We calculate then

$$<T> = \frac{T_e + T_w}{2} = \frac{3200 + 1000}{2} = 2100\,K \qquad \text{(at the throat)}$$

$$\left(\frac{T_e}{<T>}\right)^{0.8-0.2w} \simeq \left(\frac{3000}{2100}\right)^{0.61} \simeq 1.3$$

and so, using equation (30),

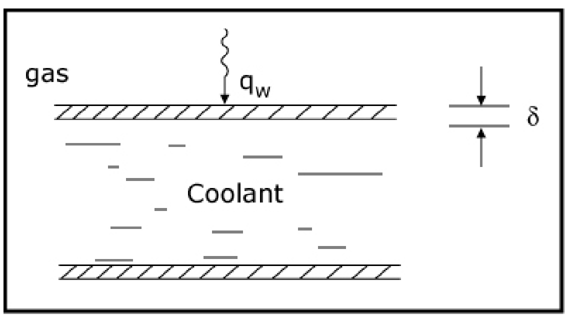
$$\underline{h_g \simeq 160,000\,\mathrm{w/m^2/K}}$$

and $q_w \simeq 160,000\left(T_{aw_t} - 1000\right)$

$$\left(T_{aw}\right)_t = T_t\left(1 + \underset{0.9}{r}\,\frac{\gamma-1}{2}\overset{1}{\cancel{\mu^2}}\right) = 1.057\times 3200 \simeq 3400\,\mathrm{K} \qquad \text{(slightly less than } T_c\text{)}$$

$$q_w \simeq 160,000\times 2400 = \underline{3.8\times 10^8\ \mathrm{W/m^2}}$$

This is a very high level of heat flux. To visualize the implications, suppose this q_w had to be transmitted through a thin metal plate (thickness δ, thermal conductivity k).



One would have $q_w = K\dfrac{\Delta T}{\delta}$ where ΔT is the temperature drop through the metal .

As an initial guess, suppose the metal were stainless steel $\left(K \simeq 20\ \mathrm{W/m/K}\right)$, and δ =1mm. Then

$$\Delta T = \frac{q_w\delta}{K} = \frac{3.8\times 10^8\times 10^{-3}}{20} = 19,000\,\mathrm{K}\ !!$$

Obviously, this is unacceptable. Try using Copper instead, with $K \simeq 400\ \mathrm{W/m/K}$ (twenty times better). This gives ΔT =950K, still not acceptable (copper would be very soft then). The plate would have to be thinner <u>and</u> made of copper. Not an easy problem.

More rationally

The S_t or h_g should depend on x, distance from start of nozzle, since the B.L. is still developing (not fully developed). In addition, there should be some accounting for

- acceleration
- property variation through B.L.
- cylindrical geometry

The article by Rubsin and Inonye (ch. 8 in Rosenhow and Hartnett's Handbook of Heat Transfer, McGraw-Hill, 1973) gives a general formula for turbulent B.L. In an cylinder, with acceleration:

$$\boxed{S_t(x) = \frac{A}{s\left(\frac{\rho_e u_e x_{eff}}{\mu_e}\right)^n F_c^{1-n} F_{R\theta}^n}} \qquad (\text{and } h_g = \rho_e u_e c_p S_t)$$

$s = \frac{c_f/2}{c_h} \simeq 1$ found walls.

$\left.\begin{matrix} A \\ n \end{matrix}\right\}$ = constants, depending on Reynolds no. based on mom. th.

$R_{e_\theta} > 4000$, $A = 0.0131$, $n = \frac{1}{7}$

$R_{e_\theta} < 4000$, $A = 0.0293$, $n = \frac{1}{5}$

$\left.\begin{matrix} F_c \\ F_{R_\theta} \end{matrix}\right\}$ = Factors for property variability. Can take several nearly equivalent forms. A simple one from Eckert, is

$$\left.\begin{matrix} F_c = \dfrac{\rho_e}{\rho\left(<T>\right)} = \dfrac{<T>}{T_e} \\ \\ F_{R\theta} = \dfrac{\mu_e}{\mu\left(<T>\right)} \simeq \left(\dfrac{T_e}{<T>}\right)^w \left(w \simeq 0.6\right) \end{matrix}\right\} \frac{<T>}{T_e} = 0.28 + 0.50\frac{T_w}{T_e} + 0.22\frac{T_{aw}}{T_e}$$

$$T_{aw} = T_e + r\frac{u_e^2}{2} = T_e\left(1 + r\frac{\gamma - 1}{2}M_e^2\right)$$

$r \simeq 0.9$ (recovery factor)

The "effective distance" x_{eff} is related to the actual distance x through an integral (accounting for memory of past acceleration)

$$x_{eff}(x) = \int_0^x \frac{f(x')}{f(x)} dx'$$

where $f = \dfrac{\rho_e u_e \left(zR\mu_e^n\right)^{\frac{1}{1-n}}}{F_c F_{R_\theta}^{\frac{n}{1-n}}}$

$$z = \frac{h_{aw} - h_w}{h_{t_e} - h_w} \quad \left(h_{aw} = h_e + r\frac{u_e^2}{2}\right)$$

R=R(x)= body radius at x.

For a quick estimate of R_{e_θ}, we can simplify further to the flat-plate case, in which

$$\frac{d\theta}{dx} = \frac{c_f}{2},$$

with $\dfrac{c_f}{2} = \begin{cases} 0.0128/R_{e_\theta}^{1/4} & \left(R_{e_\theta} < 4000\right) \\ 0.0065/R_{e_\theta}^{1/6} & \left(R_{e_\theta} > 4000\right) \end{cases}$, and with $\dfrac{d\theta}{dx} = \dfrac{dR_{e_\theta}}{dR_{e_x}}$

$$\frac{dR_{e_\theta}}{dR_{e_x}} = \begin{cases} \dfrac{0.0128}{R_{e_\theta}^{1/4}} \\ \dfrac{0.0065}{R_{e_\theta}^{1/6}} \end{cases} \rightarrow \begin{cases} \dfrac{4}{5} R_{e_\theta}^{5/4} = 0.0128\, R_{e_x} \\ \dfrac{6}{7} R_e^{7/6} = 0.0065\, R_{e_x} \end{cases} \rightarrow \boxed{R_{e_\theta} \begin{cases} 0.0366\, R_{e_x}^{4/5} \\ 0.0152\, R_{e_x}^{6/7} \end{cases}} \begin{matrix} R_{e_\theta} < 4000 \\ \left(R_{e_x} < 1.99 \times 10^1\right) \\ R_{e_\theta} > 4000 \end{matrix}$$

$$\text{or } \boxed{\frac{R_{e_\theta}}{R_{e_x}} = \frac{\theta}{x} = \begin{cases} \dfrac{0.0366}{R_{e_x}^{0.2}} \\ \dfrac{0.0152}{R_{e_x}^{1/7}} \end{cases}}$$

For large rockets, $R_{e_{x_{throat}}}$ tends to be $\sim 10^7 - 10^8$, so the high R_e formulas should be better, despite the common use of Bartz's formulae, which are based on the low R_e formulation. Fortunately, differences tend to be small, and are marked often by other uncertainties (surface films, fluid properties).

Example and Comparisons:

Consider nozzle $R = x\tan x + \dfrac{R_t^2}{4\tan\alpha}\dfrac{1}{x}$ $\qquad x = \dfrac{R \pm \sqrt{R^2 - R_t^2}}{2\tan\alpha}$

with origin at $x = x_c = \dfrac{R_c - \sqrt{R_c^2 - R_t^2}}{2\tan\alpha}$

and $\dfrac{R_c}{R_t} = 1.5$, $\alpha = 15^o$

and going through throat at $x = x_t = \dfrac{R_t}{2\tan\alpha}$

Using γ=1.25 and the $R_{e_\theta} > 4000$ option, we find

$$\left(\frac{x_{eff}}{R_t}\right)_{throat} = \int_{\frac{x_c}{R_t}}^{\frac{x_t}{R_t}} M^{5/12}\left(\frac{1.125}{1+0.125M^2}\right)^{1.875}\left(\frac{0.6979}{0.6515+0.0464M^2}\right)^{0.9} d\left(\frac{x}{R_t}\right)$$

where $\dfrac{R}{R_t} = \dfrac{x}{R_t}\tan 15^o + \dfrac{1/4}{\left(\dfrac{x}{R_t}\right)\tan 15^o} \Rightarrow \dfrac{R}{R_t}(x)$

and $\dfrac{1}{M^{1/2}}\left(\dfrac{1+0.125M^2}{1.125}\right)^{2.25} = \dfrac{R}{R_t} \Rightarrow M(x)$

The integration gives $\left(\dfrac{x_{eff}}{R_t}\right)_{throat} = 1.0892$

Compared to $\dfrac{x_t - x_c}{R_t} = 1.153$ $\left(\text{and } \dfrac{x_c}{R_t} = 0.713\right)$

Since x_{eff} appears to the $1/7$ power, the memory/acceleration effect (up to the throat) is <u>insignificant</u>.

The throat S_t is then

(0.9) $\quad (T_{aw} = 3263)$

(using r=1, $T_w = 1000\,K$, $T_c = 3300\,K$, $\frac{T_e}{t} = \frac{2}{2.25}3300 = 2933\,K$)

$$\left(S_t\right)_{throat} = \frac{0.0131}{\left(\frac{\rho u\left(x_t - x_c\right)}{\mu}\right)_{throat}^{1/7}\left(\frac{<T>}{T_e}\right)_{throat}^{0.771}}$$

$$\frac{<T>}{T_e} = 0.28 + 0.50\,\frac{1000}{2933} + 0.22\,\frac{\overset{(3263)}{3300}}{2933} = \overset{(0.6952)}{0.6979}$$

Take $P_c = 2\times10^7\,N/m^2$,

M= 25 g/mol

$$c^* = \frac{\sqrt{R_g T_c}}{\Gamma} = \frac{\sqrt{\frac{8.314}{0.025}3300}}{\sqrt{1.25}\left(\frac{2}{2.25}\right)^{\frac{2.25}{0.5}}} = 1592\,m/s$$

$$\Rightarrow \left(\rho u\right)_t = \frac{P_c}{c^*} = 12560\ Kg/s/m^2$$

$$\frac{x_t}{R_t} = \frac{1}{2\tan 15^o} = 1.866 \qquad \frac{x_c}{R_t} = \frac{1.5 - \sqrt{1.5^2 - 1}}{2\tan 15^o} = 0.7128$$

and, with $R_t = 0.3\,m$,

$$\mu \simeq 6.8\times10^{-5}\left(\frac{T}{3000}\right)^{0.6} \Rightarrow \mu_{throat} = 6.70\times10^{-5}\ Kg/m\,sec$$

One gets,

$\underline{(S_t)_{throat} = 0.00133}$ (0.00124 using x_t instead of $x_t - x_c$)

Using the $R_{e_\theta} < 4000$ option (small rockets)

$$(S_t)_{throat} = \frac{0.0293}{\left(\frac{\rho u x_{eff}}{\mu}\right)_{throat}^{0.2} \left(\frac{< T >}{T_e}\right)_{throat}^{0.68}}$$

and using again $x_{eff} = x_t - x_c$, etc,

we get $\underline{(S_t)_{throat} = 0.00102}$ (0.000933 using x_t)

For comparison, the "fully developed pipe flows" formulation would give

$$S_t = \frac{R_g}{\rho u_e c_p} = \frac{0.026}{D_t^{0.2}} \left(\frac{c^*}{P_c}\right)^{0.2} \mu_e^{0.2} \left(\frac{T_e}{< T >}\right)^{0.8-0.2w} \underbrace{\left(\frac{A_t}{A}\right)}_{1 \text{ at throat}}^{0.9}$$

$\underline{(S_t)_{throat} = 0.000958}$

This is close to the $R_\theta < 4000$ results above (and, indeed, the coefficients are for $R_\theta < 4000$). But this appears <u>coincidental</u>, based on the fact that for most nozzles, $\Delta x \sim R_t$.