16.512, Rocket Propulsion
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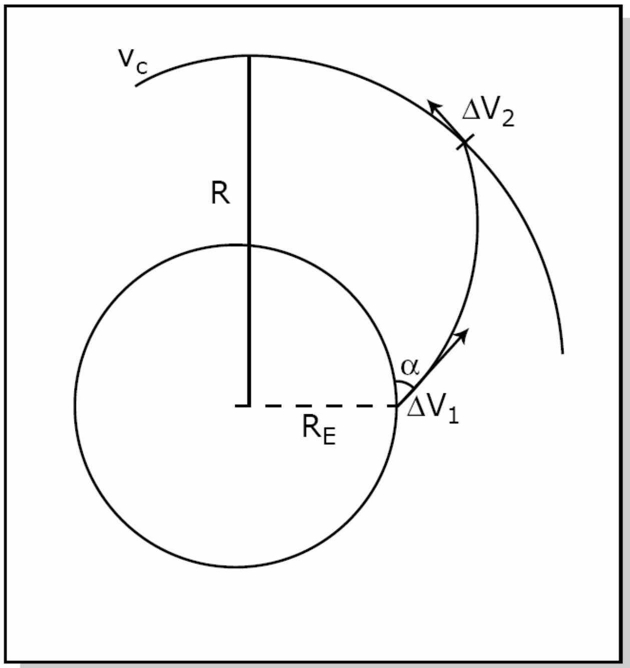
Lecture 33: Performance to LEO

ΔV Calculations for Launches to Low Earth Orbit (LEO)

Ideal Earth-to-orbit launch



$$\Delta V_1 \cos\alpha\, R_E = v_2^{'} R$$

$$\frac{\Delta V_1^2}{2} - \frac{\mu}{R_E} = \frac{V_2^{'2}}{2} - \frac{\mu}{R}$$

$$= \frac{1}{2}\left(\Delta V_1 \cos\alpha \frac{R_E}{R}\right)^2 - \frac{\mu}{R}$$

$$\Delta V_1^2 = 2\frac{\frac{\mu}{R_{\in}} - \frac{\mu}{R}}{1-\left(\frac{R_{\in}}{R}\right)^2 \cos^2\alpha}$$

$$\Delta V_1 = \sqrt{2\,\frac{\mu}{R_E}\,\frac{1-\frac{R_E}{R}}{1-\left(\frac{R_E}{R}\right)^2 \cos^2\alpha}}$$

$$v_c = \sqrt{\frac{\mu}{R}}$$

$$\Delta V_2 = v_c - v'_2 = \sqrt{\frac{\mu}{R}} - \frac{R_{\in}}{R}\cos\alpha\sqrt{2\,\frac{\mu}{R_{\in}}\,\frac{1-\frac{R_E}{R}}{1-\left(\frac{R_E}{R}\right)^2\cos^2\alpha}} = \sqrt{\frac{\mu}{R_E}}\left[\sqrt{\frac{R_E}{R}} - \frac{R_E}{R}\cos\alpha\sqrt{2\,\frac{1-\frac{R_E}{R}}{1-\left(\frac{R_E}{R}\right)^2\cos^2\alpha}}\right]$$

$$\Delta V = \Delta V_1 + \Delta V_2 \qquad \frac{\Delta V}{\sqrt{\mu / R_{\in}}} = \sqrt{\frac{2(1-\eta)}{1-\eta^2\cos^2\alpha}} + \sqrt{\eta} - \eta\cos\alpha\sqrt{2'\,\frac{1-\eta}{1-\eta^2\cos^2\alpha}}$$

$$\frac{R_{\in}}{R} = \eta \qquad \boxed{\frac{\Delta V}{\sqrt{\mu / R_{\in}}} = \sqrt{2(1-\eta)\frac{1-\eta\cos\alpha}{1+\eta\cos\alpha}} + \sqrt{\eta}} \qquad \text{(increasing f. of } \alpha\text{)}$$

For $\alpha = 0$

$$\boxed{\frac{\Delta V_{MIN}}{\sqrt{\mu / R_E}} = \sqrt{2\frac{(1-\eta)^2}{1+\eta}} + \sqrt{\eta}}$$

Note: Max at $\eta = 0.064178 \rightarrow R = 99,260\,km$ (worst altitude)

$$\left(\frac{\Delta V_{MIN}}{\sqrt{\mu / R_F}} \right)_{MAX} = 1.5363$$

$$\left(\sqrt{\frac{\mu}{R_\in}} = 7910\, m/s \right)$$

$$\frac{1}{\eta} - 1 = \varepsilon \qquad \eta = \frac{1}{1+\varepsilon} \qquad 1-\eta = \frac{\varepsilon}{1+\varepsilon}$$

$$1+\eta = \frac{2+\varepsilon}{1+\varepsilon}$$

$$\simeq \sqrt{2\frac{\varepsilon^2}{(1+\varepsilon)^2}\frac{1+\varepsilon}{1+\varepsilon/2}} + \sqrt{\frac{1}{1+\varepsilon}} = 1 - \frac{\varepsilon}{2} + \frac{3}{8}\varepsilon^2 \ldots + \varepsilon - \frac{3}{4}\varepsilon^2 = 1 + \frac{\varepsilon}{2} - \frac{3}{8}\varepsilon^2 \ldots$$

$$\varepsilon\left(1 - \frac{3}{4}\varepsilon \ldots\right) \qquad \alpha = 0 \qquad \eta = 0.9 \rightarrow 1.05128$$

(approx. 1.05093)

$\Delta V / \sqrt{\mu / R_\in}$ (ΔV)	$\eta = 0.15095$ (GEO, R=42,200Km)	$\eta = 0.23951$ $\left(\frac{1}{2} day, \eta = 26,580\,Km\right)$	$\eta = 0.87620$ $(Z = 900\,Km)$	$\eta = 0.91392$ $(Z = 600\,Km)$	$\eta = 0.95502$ $(Z = 300\,Km)$
$\alpha = 0^0$	1.50775 (11,918 m/s)	1.45534 (11,504 m/s)	1.06387 (8,409 m/s)	1.04399 (8252 m/s)	1.02275 (8,084 m/s)
$\alpha = 15^0$	1.51366 (11,965 m/s)	1.46373 (11,570 m/s)	1.07960 (8,534 m/s)	1.05952 (8375 m/s)	1.03748 (8201 m/s)
$\alpha = 30^0$	1.53109 (12,102 m/s)	1.48854 (11,766 m/s)	1.12032 (8,856 m/s)	1.09755 (8676 m/s)	1.06953 (8454 m/s)

NOTE: $\frac{\Delta V_1}{\sqrt{\mu / R_E}} \simeq 1 + \frac{\varepsilon}{4} - \frac{5\varepsilon^2}{32} \qquad \frac{\Delta V_2}{\sqrt{\mu / R_E}} \simeq \frac{\varepsilon}{4} - \frac{7\varepsilon^2}{32}$

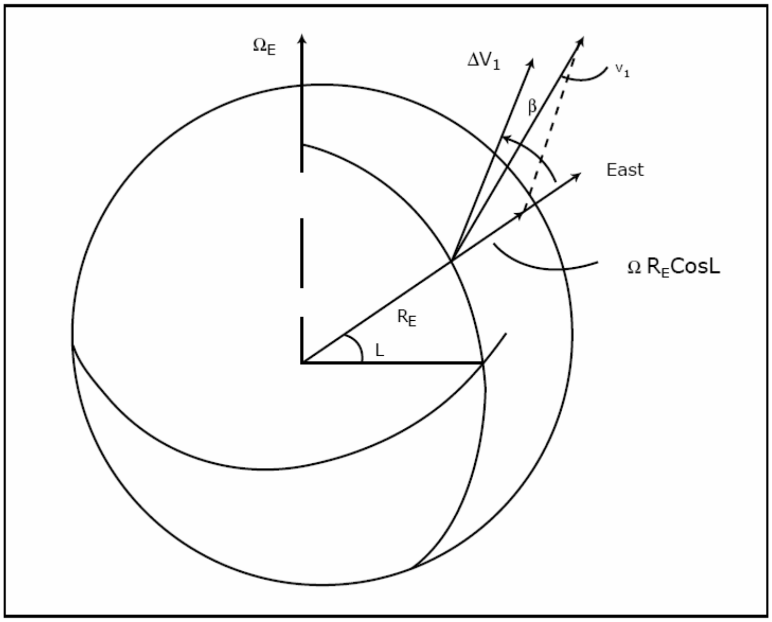
So, for LEO, mainly ΔV_1 (apogee kick)

Sticking to $\alpha = 0$, variation with R

	$\frac{1}{\eta} = \frac{R}{R_\epsilon} = 1.05$	1.1	1.05	2	4	6	10	15.6
$\frac{\Delta V}{\sqrt{\mu / R_\epsilon}}$	1.02041	1.04651	1.18164	1.28446	1.44868	1.49934	1.52978	1.53626

Effects of Earth's Rotation

(a) ΔV reduction

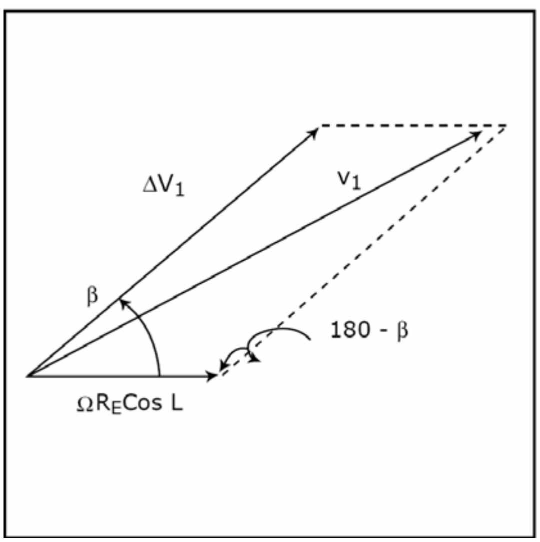


$\Omega R_\epsilon = 463\, m/\sec$

β is launch azimuth w.r.t East

($\alpha = 0$, near-horizontal launch)

ΔV_1 = rocket-imparted ΔV



Starting velocity (abs.) is now

$$v_1 = \sqrt{(\Delta V_1)^2 + (\Omega R_\epsilon \cos L)^2 + 2\Delta V_1 (\Omega R_\epsilon \cos L) \cos \beta}$$

So, v_1 replaces ΔV_1 in previous formulation

$$v_1 = \sqrt{2\frac{\mu}{R_E}\frac{1-\frac{R_E}{R}}{1-\left(\frac{R_E}{R}\right)^2 cos^2\alpha}} \xrightarrow[\alpha=0]{} \sqrt{\frac{2\frac{\mu}{R_E}}{1+\frac{R_E}{R}}} = \sqrt{\frac{\mu R / R_E}{\left(\frac{R_E+R}{2}\right)}}$$

$$\frac{2\mu\frac{R}{R_E}}{R+R_E} = \left(\Delta V_1\right)^2 + \left(\Omega R_\in cos\, L\right)^2 + 2\Delta V_1\left(\Omega R_E\, cos\, L\right) cos\, \beta$$

$$\Delta V_1 = -\left(\Omega R_E\, cos\, L\right) cos\, \beta + \sqrt{\left(\Omega R_\in cos\, L\, cos\, \beta\right)^2 + \frac{2\mu\frac{R}{R_E}}{R+R_E} - \left(\Omega R_E\, cos\, L\right)^2}$$

$$\boxed{\Delta V_1 = -\Omega R_E\, cos\, L\, cos\, \beta + \sqrt{\frac{2\mu\frac{R}{R_\in}}{R+R_\in} - \left(\Omega R_E\, cos\, L \sin\beta\right)^2}}$$

Notice rotation reduces ΔV_1 <u>even for $\beta = 90^0$</u>. The benefit is low for some larger β (Westwards launch). For $v_1 = \Delta V_1$, need

$$cos\, \beta = -\frac{\Omega R_\in cos\, L}{2\Delta V_1} \simeq -0.056\, cos\, L \qquad \text{(for } \Delta V_1 = 8200\, m/s\text{)}$$

(for $L = 28.5^0$, $\beta = 92.8^0$)

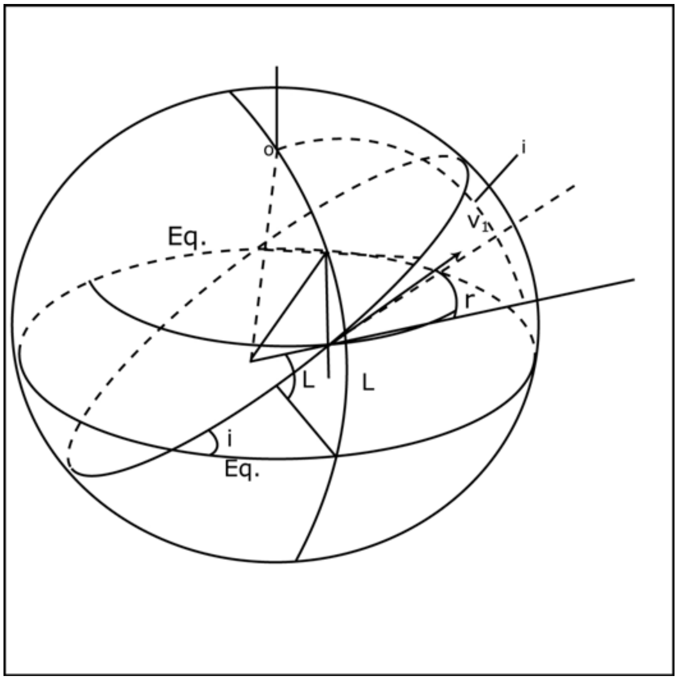
<u>Example</u>: For $L = 28.5^0$, R = 6370+500 = 6870 Km,

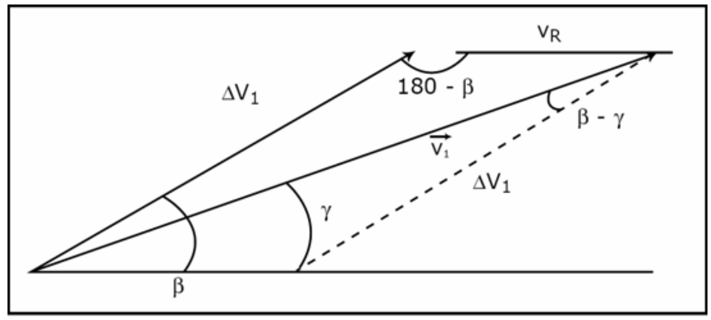
$$\Delta V_1 = -407\, cos\, \beta + \sqrt{\frac{6.48399\times 10^7}{(8052.8)^2} - \left(407\, sin\, \beta\right)^2}$$

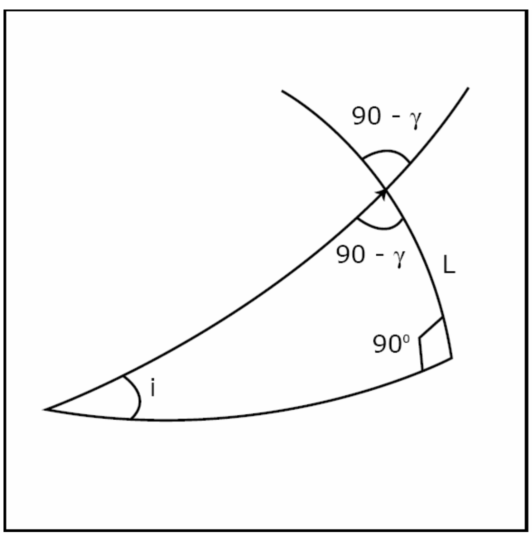
$\beta\left(^0\right)$	0	$\pm 30^0$	$\pm 60^0$	$\pm 90^0$	$\pm 120^0$	$\pm 150^0$	$\pm 180^0$
ΔV_1 (m/s)'	7645 m/s	7694	7841	8042	8248	8402	8459
ΔV_1 reduction	407 m/s	355 m/s	211 m/s	10 m/s	-195 m/s	-350 m/s	-407 m/s

(b) Orbit inclination

For $\beta = 0$ (launch due East), i = L. For any other azimuth, higher inclination.







$$v_R = \Omega R_\epsilon \cos L$$

$$\frac{v_R}{\sin(\beta - \gamma)} = \frac{\Delta V_1}{\sin\gamma}$$

$$v_R \frac{\sin\gamma}{\cos\gamma} = \Delta V_1 \left(\sin\beta \cancel{\cos\gamma} - \cos\beta \frac{\sin\gamma}{\cos\gamma} \right)$$

$$\tan\gamma = \frac{\Delta V_1 \sin\beta}{\Delta V_1 \cos\beta + v_R}$$

$$\boxed{\tan\gamma = \frac{\tan\beta}{1 + \left(\dfrac{v_R}{\Delta V_1 \cos\beta} \right)}}$$

Given two angles and the side included, find opposite angle

$$\boxed{\cos i = -\cos\left(90^0 - \gamma\right)\cos 90^0 + \sin\left(90 - \gamma\right)\sin 90 \cos L = \cos\gamma \cos L}$$

Example: Continuing from previous example,

$L = 28.5^0 \rightarrow v_R = 407\, m/s,\ 'R = 500\, Km \rightarrow \mathrm{R_E}$

$$\tan\gamma = \frac{\tan\beta}{1 + \dfrac{407}{\Delta V_1 \cos\beta}}, \qquad \cos i = 0.87882 \cos\gamma$$

ΔV_1 from previous table

β	0	30^0	60^0	90^0	120^0	150^0	180^0
α	0	28.54^0	57.49^0	87.10^0	117.49^0	148.55^0	180^0
i	28.5^0	39.5^0	61.8^0	87.5^0	113.9^0 (66.1^0)	(41.4^0)	(28.5^0)

retro-orbits (westwards)

-30^0	-60^0	-90^0
-28.50^0	-57.39^0	
39.4^0	61.7^0	

slightly different inclination

In reality, we probably require the orbit altitude, the orbit inclination and then launch azimuth β must be calculated

$$\cos\gamma = \frac{\cos i}{\cos L} \qquad \gamma = \cos^{-1}\left(\frac{\cos i}{\cos L}\right) \qquad 1 + \tan^2\gamma = \frac{1}{\cos^2\gamma} = \frac{\cos^2 L}{\cos^2 i}$$

$$\tan\gamma = \sqrt{\frac{\cos^2 L}{\cos^2 i} - 1} = \frac{\Delta V_1 \sin\beta}{\Delta V \cos\beta + v_R}$$

$$\sin\gamma = \sqrt{1 - \frac{\cos^2 i}{\cos^2 L}}$$

with $\Delta V_1 = \sqrt{\dfrac{2\mu \dfrac{R}{R_\epsilon}}{R + R_\epsilon} - v_R^2 \sin^2\beta - v_R \cos\beta}$

$$\Delta V_1 \cos\beta \tan\gamma + v_R \tan\gamma = \Delta V_1 \sin\beta$$

or $\left(\Delta V_1\right)^2 + v_R^2 + 2v_R\Delta V_1 \cos\beta = \dfrac{2\mu\dfrac{R}{R_E}}{R+R_E}$ $\qquad$ $\Delta V_1 = \dfrac{v_R\tan\gamma}{\sin\beta - \cos\beta\tan\gamma} = \dfrac{v_R}{\dfrac{\sin\beta}{\tan\gamma} - \cos\beta}$

$$\frac{v_R^2}{\left(\dfrac{\sin\beta}{\tan\gamma} - \cos\beta\right)^2} + v_R^2 + \frac{2v_R^2\cos\beta}{\dfrac{\sin\beta}{\tan\gamma} - \cos\beta} = \frac{2\mu\dfrac{R}{R_E}}{R+R_E}$$

$$v_R^2\left[1 + \left(\frac{\sin\beta}{\tan\gamma} - \cos\beta\right)^2 + 2\cos\beta\left(\frac{\sin\beta}{\tan\gamma} - \cos\beta\right)\right] = \frac{2\mu\dfrac{R}{R_E}}{R+R_E}\left(\frac{\sin\beta}{\tan\gamma} - \cos\beta\right)^2$$

$$1 + \frac{\sin^2\beta}{\tan^2\gamma} + \cos^2\beta - \cancel{\frac{2\sin\beta\cos\beta}{\tan\gamma}} + \cancel{\frac{2\sin\beta\cos\beta}{\tan\gamma}} - 2\cos^2\beta$$

$$\sin^2\beta\left(1 + \frac{1}{\tan^2\gamma}\right)$$

$$\frac{\sin^2\beta}{\sin^2\gamma} \qquad\qquad v_R^2 = \frac{2\mu\dfrac{R}{R_E}}{R+R_E}\sin^2\gamma\left(\frac{1}{\tan\gamma} - \frac{1}{\tan\beta}\right)^2$$

$$\boxed{\tan\beta = \frac{1}{\dfrac{1}{\tan\gamma} - \dfrac{v_R}{\sin\gamma}\sqrt{\dfrac{R+R_E}{2\mu\dfrac{R}{R_E}}}}}$$

$$\tan\beta = \frac{\tan\gamma}{1 - \dfrac{v_R}{\cos\gamma}\sqrt{\dfrac{R+R_E}{2\mu\dfrac{R}{R_E}}}}$$

<u>Check</u>: $\left.\begin{matrix} L = 28.5^0 \\ i = 61.8^0 \end{matrix}\right) \rightarrow \gamma = 57.47^0 \rightarrow \quad \tan\beta = \dfrac{1}{0.6377 - \left(\dfrac{407}{0.8431\times 8053}\right)^0} = 1.7308$

$\beta = 59.98^0 \left(60^0\right)$

$$\tan\beta = \frac{\sqrt{\dfrac{cos^2 L}{cos^2 i} - 1}}{1 - v_R \dfrac{cos\,L}{cos\,i}\sqrt{\dfrac{R + R_E}{2\mu \dfrac{R}{R_E}}}}$$

$$\boxed{\tan\beta = \frac{\sqrt{cos^2 L - cos^2 i}}{cos\,i - \dfrac{v_R\,cos\,L}{\left(\sqrt{\dfrac{R + R_E}{2\mu \dfrac{R}{R_E}}}\right) - 1}}}$$

Directly: $\dfrac{V_1}{sin\,\beta} = \dfrac{v_R}{sin\left(\beta - \alpha\right)}$

$$\frac{v_R}{V_1} = cos\,\gamma - \frac{sin\,\gamma}{\tan\beta}$$

$$\tan\beta = \frac{sin\,\gamma}{cos\,\gamma - \dfrac{v_R}{V_1}} = \frac{\sqrt{1 - cos^2\gamma}}{cos\,\gamma - \dfrac{v_R}{V_1}}$$

and $\qquad cos\,\gamma = \dfrac{cos\,i}{cos\,L}$

$$Cos^2\beta = \frac{1}{1 + \tan^2\beta} = \frac{1}{1 + \dfrac{Cos^2 L - Cos^2 i}{\left(Cos\,i - \dfrac{v_R\,Cos\,L}{\left(\sqrt{}\right)}\right)^2}} = \frac{Cos\,i - \dfrac{v_R\,Cos\,L}{\left(\sqrt{}\right)}}{\left(\dfrac{v_R\,Cos\,L}{\left(\sqrt{}\right)}\right)^2 - 2\,cos\,i\,\dfrac{v_R\,cos\,L}{\left(\sqrt{}\right)} + cos^2 L}$$

$$\tan\beta = \frac{\tan\alpha}{1 - \dfrac{v_R}{cos\,\gamma}\sqrt{\dfrac{R + R_E}{2\mu R / R_E}}} \qquad \frac{1}{cos\,\beta} = \sqrt{1 + \tan^2\beta} = \sqrt{1 + \frac{\tan^2\gamma}{\left(1 - \dfrac{v_R}{cos\,\gamma}\sqrt{\dfrac{R + R_\in}{2\mu R / R_\in}}\right)^2}}$$

$$\Delta V_1 = \frac{v_R}{\cos\beta}\ \frac{1}{\dfrac{\tan\beta}{\tan\gamma} - 1}$$

$$\Delta V_1 = \frac{v_R\ cos\ \gamma}{v_R\sqrt{\dfrac{R + R_\in}{2\mu R\ /\ R_E}}}\sqrt{\left(1 - \frac{v_R}{cos\ \gamma}\sqrt{\frac{R + R_E}{2\mu R\ /\ R_E}}\right)^2 + \tan^2\gamma}$$

$$= \frac{\cancel{v_R}\ cos\ \gamma}{\cancel{v_R}\sqrt{\dfrac{R + R_E}{2\mu}}}\sqrt{\frac{1}{cos^2\gamma} + \frac{v^2{}_R}{cos^2\gamma}\left(\frac{R + R_E}{2\mu R\ /\ R_E}\right) - 2\,\frac{v_R}{cos\ \gamma}\sqrt{\left(\frac{R + R_E}{2\mu R\ /\ R_E}\right)}}$$