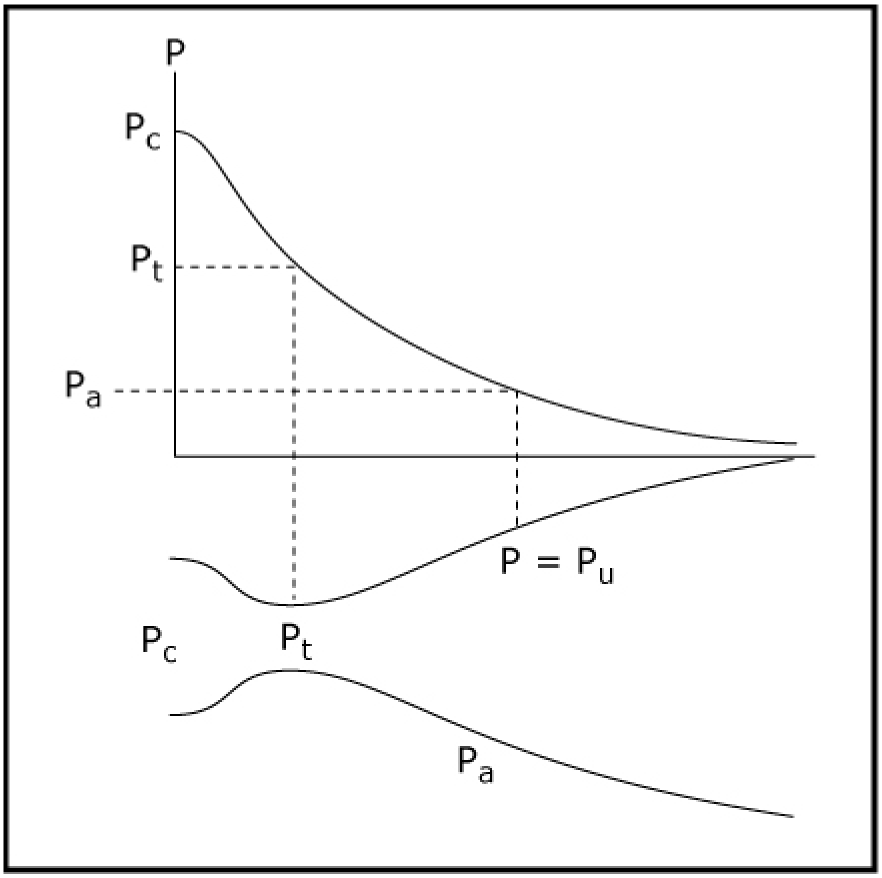
Lecture 3: Ideal Nozzle Fluid Mechanics

Ideal Nozzle Flow with No Separation (1-D)



- Quasi 1-D (slender) approximation
- Ideal gas assumed

$$F = \dot{m} u_e + \left(P_e - P_a\right) A_e$$

$$\boxed{C_F \equiv \frac{F}{P_c A_t}}$$

Optimum expansion: $P_e = P_a$

- For less $\frac{A_e}{A_t}$, $P_e > P_a$, could derive more forward push by additional expansion

- For more $\frac{A_e}{A_t}$, $P_e < P_a$, and the extra pressure forces are a <u>suction</u>, backwards

Compute $\dot{m} = \rho u A$ at <u>sonic</u> throat:

$$\dot{m} = \rho_c \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \sqrt{\gamma R_g T_c \left(\frac{2}{\gamma+1}\right)}\; A_t \quad = \underbrace{\sqrt{g}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}_{\text{call } \Gamma \approx 2/3} \frac{P_c A_t}{\sqrt{R_g T_c}} \qquad ; R_g = \frac{R}{M}$$

call $c^* = \frac{\sqrt{R_g T_c}}{\Gamma(\gamma)}$ ("characteristic velocity") $\rightarrow$ $\boxed{\dot{m} = \frac{P_c A_t}{c^*}}$

Can express u_e, P_e, A_e, etc in terms of either $\underline{M_e}$ or $\underline{\left(\frac{P_e}{P_c}\right)}$ or $\underline{\frac{A_e}{A_t}}$:

$$\frac{P_e}{P_c} = \frac{1}{\left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\gamma/\gamma-1}} \; ;$$

$$\frac{A_e}{A_t} = \left(\frac{P_t}{P_e}\right)\left(\frac{u_t}{u_e}\right) = \left(\frac{P_t}{P_e}\right)\frac{1}{M_e}\sqrt{\frac{T_t}{T_e}} = \frac{1}{M_e}\left(\frac{1 + \frac{\gamma-1}{2}M_e^2}{\frac{\gamma+1}{2}}\right)^{\frac{1}{2}+\frac{1}{\gamma-1}}$$

(exponent: $\frac{1}{2}+\frac{1}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}$)

$$\frac{P_e}{P_c} = \left(\frac{T_e}{T_c}\right)^{\gamma/\gamma-1},$$

and $\frac{T_e}{T_c} = \frac{1}{1 + \frac{\gamma-1}{2}M_e^2}$,

<u>Because</u> $c_p T_e + \frac{u_e^2}{2} = c_p T_c \rightarrow \frac{\gamma}{\gamma-1} R_g T_e + \frac{M_e^2}{2}\gamma R T_e = \frac{\gamma}{\gamma-1} R_g T_c$

$$C_F = \frac{\dot{m}}{P_c A_t} u_e + \left(\frac{P_e - P_a}{P_c}\right)\frac{A_e}{A_t} = \frac{u_e}{c^*} + \left(\frac{P_e}{P_c} - \frac{P_a}{P_c}\right)\frac{A_e}{A_t}$$

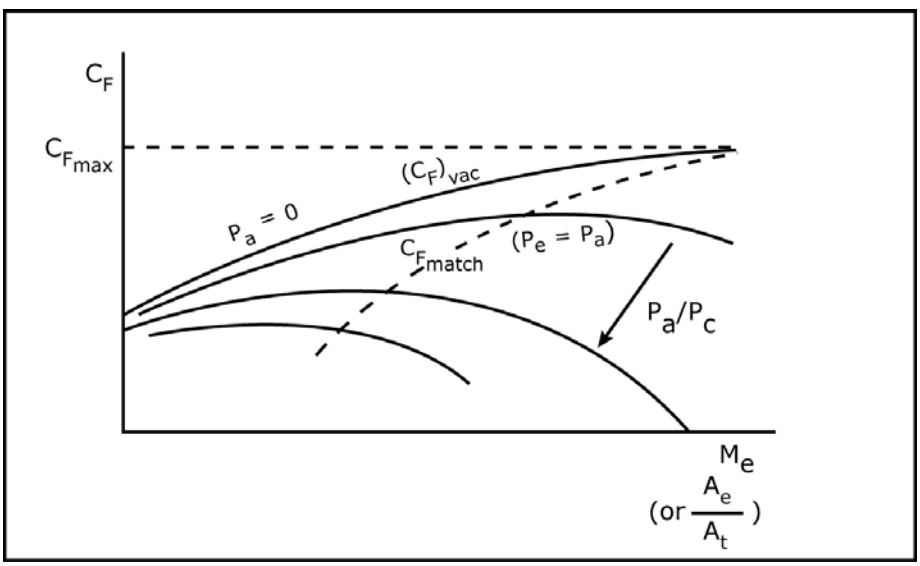
$$\frac{u_e}{c^*} = \frac{M_e\sqrt{\gamma \cancel{R_g}\,\dfrac{\cancel{T_c}}{1+\dfrac{\gamma-1}{2}M_e^2}}}{\dfrac{\sqrt{\cancel{R_g}\cancel{T_c}}}{\Gamma}} = \gamma\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{M_e}{\sqrt{1+\dfrac{\gamma-1}{2}M_e^2}}$$

In vacuum,

$$\left(P_a = 0\right)$$

$$C_{F_v} = \frac{u_e}{c^*} + \frac{P_e}{P_c}\frac{A_e}{A_t} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{\gamma M_e}{\sqrt{1+\dfrac{\gamma-1}{2}M_e^2}} + \frac{1}{M_e}\frac{\left(1+\dfrac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma+1}{2(\gamma-1)}-\frac{\gamma}{\gamma-1}}}{M_e\left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

$$\frac{\gamma+1}{2(\gamma-1)}-\frac{\gamma}{\gamma-1} = -\frac{1}{2}$$



$$\boxed{\left(C_F\right)_v = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}\frac{\gamma M_e + \dfrac{1}{M_e}}{\sqrt{1+\dfrac{\gamma-1}{2}M_e^2}}}$$

and otherwise,

$$C_F = C_{F_v} - \left(\frac{A_e}{A_t}\right)_{M_e} \frac{P_a}{P_c}$$

Note:

For $P_e = P_a$,

$$\left(C_F\right)_{\text{Matched}} = \frac{u_e}{c^*} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{\gamma M_e}{\sqrt{1+\frac{\gamma-1}{2}M_e^2}}$$

For $P_e = P_a = 0$ $\quad \left(C_F\right)_{\text{Max,Vac}} = \gamma\sqrt{\frac{2}{\gamma-1}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$

Choice of Optimum Expansion For a Rocket Flying Through an Atmosphere (P_a varying)

The thrust coefficient $C_F = \dfrac{F}{P_c A_t}$ was derived in class in the form

$$C_F = C_{F_{vac}} - \frac{P_a}{P_c}\left(\frac{A_e}{A_t}\right) \qquad (1)$$

and we also found

$$C_{F_{vac}} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{\gamma M_e + \frac{1}{M_e}}{\sqrt{1+\frac{\gamma-1}{2}M_e^2}} \qquad (2)$$

$$\frac{A_e}{A_t} = \frac{1}{M_e}\left(\frac{1+\frac{\gamma-1}{2}M_e^2}{\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (3)$$

The thrust-derived velocity increment is

$$\Delta V_F = \int_0^{t_b} \frac{F}{m}dt = P_c A_t \int_0^{t_b} \frac{C_F}{m}dt \qquad (4)$$

where $C_F = C_F(t)$ due only to the variation of P_a in (1), while $m = m(t)$ because of mass burnout. The quantities $C_{F_{vac}}$ and $\frac{A_e}{A_t}$ depend on M_e (or nozzle geometry), but are time-invariant. Substituting (1), (2) and (3) into (4),

$$\Delta V_F = P_c A_t \left[C_{F_{vac}} \int_0^{t_b} \frac{dt}{m} - \left(\frac{A_e}{A_t}\right) \int_0^{t_b} \frac{P_a}{P_c} \frac{dt}{m} \right]$$

or

$$\frac{\Delta V_F}{P_c A_t \int_0^{t_b} \frac{dt}{m}} = C_{F_{vac}} - \frac{\int_0^{t_b} \frac{P_a}{P_c} \frac{dt}{m}}{\int_0^{t_b} \frac{dt}{m}} \frac{A_e}{A_t} \tag{5}$$

We now make the approximation that the trajectory will change little when we vary M_e (and hence $C_{F_{vac}}$, $\frac{A_e}{A_t}$). We can then regard the time integrals in (5) as fixed quantities while we optimize M_e. Define the non-dimensional variables

$$v = \frac{\Delta V_F}{P_c A_t \int_0^{t_b} \frac{dt}{m}} \quad ; \quad p = \frac{\int_0^{t_b} \frac{P_a}{P_c} \frac{dt}{m}}{\int_0^{t_b} \frac{dt}{m}} \tag{6}$$

so that (5) becomes

$$v = C_{F_{vac}}(M_e) - p\left(\frac{A_e}{A_t}\right)(M_e) \tag{7}$$

and we can now differentiate v w.r.t M_e (holding p=const.)

$$\frac{\partial v}{\partial M_e} = \frac{\partial C_{F_{vac}}}{\partial M_e} - p \frac{\partial \left(A_e / A_t \right)}{\partial M_e} = 0 \tag{8}$$

From (2) and (3), the factor $\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$ appears in both terms of (8) and can be ignored. We then have

$$\frac{\partial}{\partial M_e}\left(\frac{\gamma M_e + \frac{1}{M_e}}{\sqrt{1+\frac{\gamma-1}{2}M_e^2}}\right) = p\frac{\partial}{\partial M_e}\left[\frac{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{M_e}\right]$$

$$\frac{\gamma - \frac{1}{M_e^2}}{\sqrt{1+\frac{\gamma-1}{2}M_e^2}} - \left(\gamma M_e + \frac{1}{M_e}\right)\frac{1}{2}\frac{\frac{\gamma-1}{2}2M_e}{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{3/2}} = p\frac{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{M_e}\left[\frac{\gamma+1}{2(\gamma-1)}\frac{\frac{\gamma-1}{2}2M_e}{1+\frac{\gamma-1}{2}M_e^2} - \frac{1}{M_e}\right]$$

Multiply times $\left(1+\frac{\gamma-1}{2}M_e^2\right)^{3/2}$, and note that $\frac{\gamma+1}{2(\gamma-1)}+\frac{1}{2}=\frac{\gamma}{\gamma-1}$

$$\left(\gamma-\frac{1}{M_e^2}\right)\left(1+\frac{\gamma-1}{2}M_e^2\right) - \frac{\gamma-1}{2}\left(\gamma M_e^2+1\right) = p\frac{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\gamma/\gamma-1}}{M_e^2}\left[(\gamma+1)M_e^2 - \left(1+\frac{\gamma-1}{2}M_e^2\right)\right]$$

Expand & simplify

$$\underbrace{\gamma + \frac{\gamma(\gamma-1)}{2}M_e^2 - \frac{1}{M_e^2} - \frac{\gamma-1}{2} - \frac{\gamma(\gamma-1)}{2}M_e^2 - \frac{\gamma-1}{2}}_{1-\frac{1}{M_e^2}=\frac{M_e^2-1}{M_e^2}} = p\frac{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\gamma/\gamma-1}}{M_e^2}\left(M_e^2-1\right)$$

Cancel the factor $\frac{M_e^2-1}{M_e^2}$ ($M_e = 1$ is clearly not an optimum!)

$$1 = p\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\gamma/\gamma-1}$$

or

$$1+\frac{\gamma-1}{2}M_e^2=\left(\frac{1}{p}\right)^{\frac{\gamma-1}{\gamma}} \qquad \boxed{M_{e_{OPT}}=\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{1}{p}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}} \tag{9}$$

Notice that the exit pressure is given by

$$\frac{P_e}{P_c}=\frac{1}{\left(1+\frac{\gamma-1}{2}M_e^2\right)^{\gamma/\gamma-1}} \tag{10}$$

and so the optimum exit pressure turn out to be

$$\boxed{\left(\frac{P_e}{P_c}\right)_{OPT}=p} \tag{11}$$

However, if $p<0.4\frac{P_{a_o}}{P_c}$, this would imply $P_e<0.4P_{a_o}$, and there would be flow separation at the highest P_a (on the ground). To avoid this, the optimality condition must be amended to

$$\boxed{\left(\frac{P_e}{P_c}\right)_{OPT}=\text{Greater of }\left\{\rho,\ 0.4\frac{P_{a_o}}{P_c}\right\}} \tag{12}$$

with a similar expression for M_e:

$$\boxed{M_{e_{OPT}}=\text{Least of}\left\{\sqrt{\frac{2}{\gamma-1}\left[\left(\frac{1}{p}\right)^{\frac{\gamma-1}{\gamma}}-1\right]},\quad \sqrt{\frac{2}{\gamma-1}\left[\left(2.5\frac{P_c}{P_{a_o}}\right)^{\frac{\gamma-1}{\gamma}}-1\right]}\right\}} \tag{13}$$

The limiting condition in which the whole burn occurs at P_{a_o} is simple.

We then obtain

$$p=\frac{\int_0^{t_b}\frac{P_{a_o}}{P_c}\frac{dt}{m}}{\int_0^{t_b}\frac{dt}{m}}=\frac{P_{a_o}}{P_c} \tag{14}$$

and the optimality condition (12) yields $\underline{(P_e)_{OPT} = P_{a_o}}$, i.e., the nozzle should be <u>pressure-matched</u>, as expected.

As more and more of the burn shifts to higher altitudes, p decreases from $\frac{P_{a_o}}{P_c}$. As long as it still remains above $0.4\frac{P_{a_o}}{P_c}$, equation (11) gives some intermediate optimum design, and if p drops below $0.4\frac{P_{a_o}}{P_c}$, the nozzle should be designed to be on the verge of separation on the ground.

Nozzle Flow Separation Effects

Rule of thumb (to be explored later):

<u>Flow separates at the point in the nozzle where</u>

$\underline{P \approx 0.4P_a}$ (Summerfield criterion)

So, if $P_e > 0.4P_a$ (even if $P_e < P_a$), <u>no separation</u>

After separation, roughly parallel flow, at $P = P_a$ (no strong p gradients in "dead water" region to turn flow).

So zero thrust contribution ⟶ <u>Performance with separation at that of a nozzle with exit pressure</u> $\underline{P_e^{'} = 0.4P_a}$

So,

(a) $P_a < \frac{P_e\text{(full nozzle)}}{0.4}$,

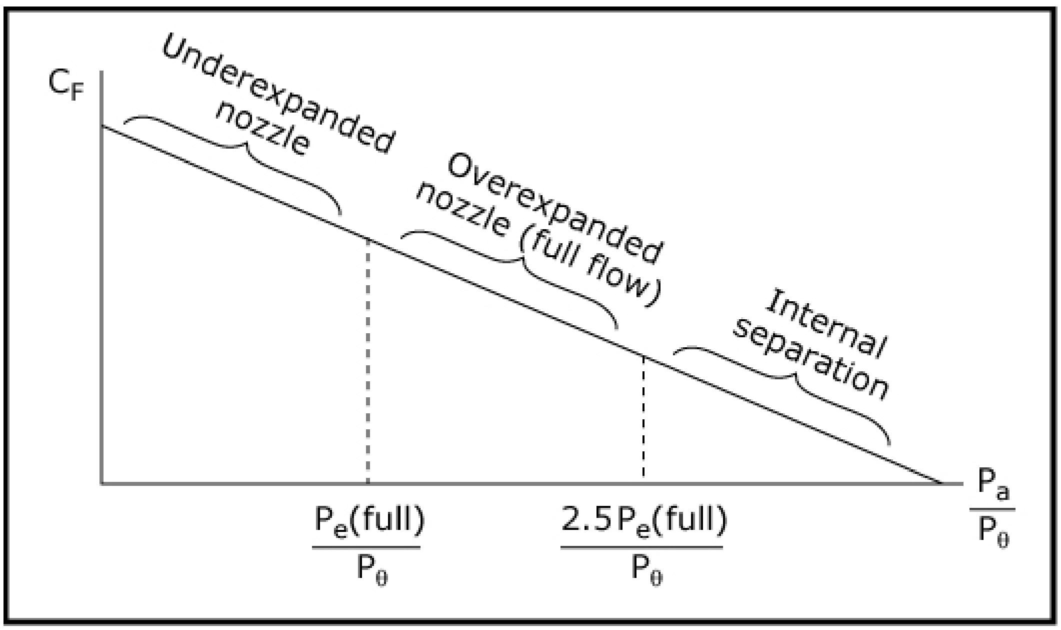
$$\boxed{C_F = C_{F_{\text{vac}}} - \frac{P_a}{P_o}\frac{A_e}{A_t}}$$

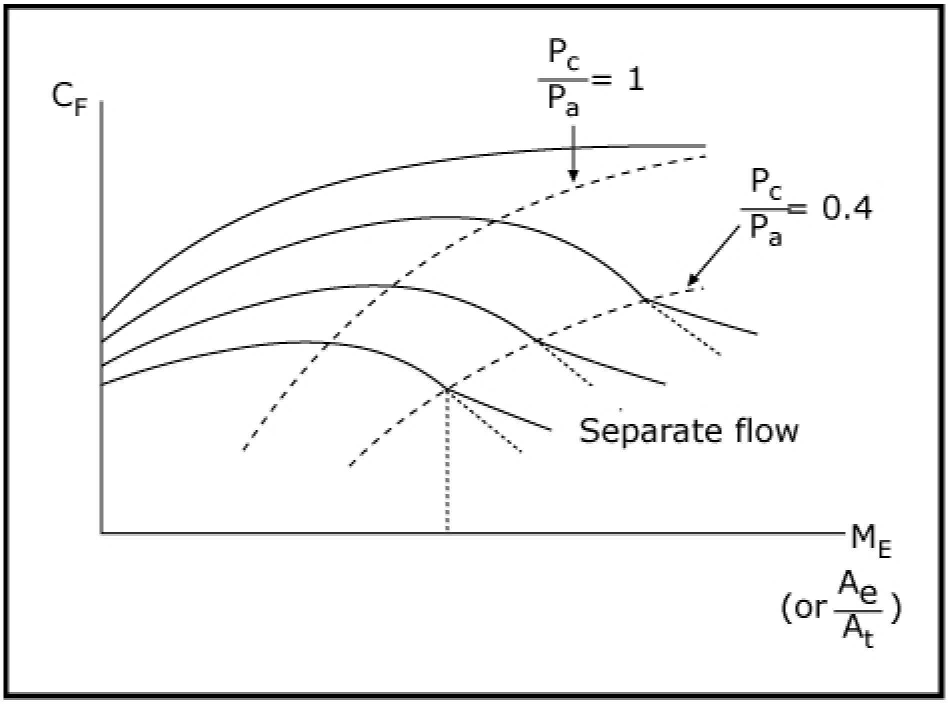
where $C_{F_{\text{vac}}}$ is $f(M_e)$ and $\frac{A_e}{A_t}$ is $g(M_e)$

(b) $P_a > \dfrac{P_e(\text{full nozzle})}{0.4}$,

calculate $\begin{cases} M_e^{'} = M\left(P_e^{'} = 0.4\,P_a\right) \\ \dfrac{A_e^{'}}{A_t} = \dfrac{A_e^{'}}{A_t}\left(M_e^{'}\right) \end{cases}$

then $\boxed{C_F = C_{F_{\text{vac}}}\left(M_e^{'}\right) - \dfrac{P_a}{P_o}\dfrac{A_e^{'}}{A_t}}$




C_F
$\frac{P_c}{P_a} = 1$
$\frac{P_c}{P_a} = 0.4$
Separate flow
M_E
(or $\frac{A_e}{A_t}$)