16.50 Lecture 7

Subject: Modeling of rocket nozzles; effects of nozzle area ratio.

In the last lecture we saw how the throat area of the nozzle controls the mass flow rate. Now we will explore the effects of the shape of the nozzle downstream of the throat.

The Mach number and hence velocity at any point in the nozzle is determined by the ratio of the area of the stream tube to the area of the throat (the area ratio), so if we <u>assume</u> that

a) The nozzle flows full, i.e. the streamtube shape matches the shape of the nozzle
b) The flow is supersonic to the point x downstream of the throat,

then we can find M(x) from $A(x)/A_t$ and it in turn determines p(x), T(x) and u(x).

As we shall see, it is the <u>pressure</u> that determines whether the nozzle flows full, so it is convenient to relate the exit velocity directly to the pressure. We do this from the Energy Equation,

$$T_c \equiv T + \frac{u^2}{2c_p}$$

$$u^2 = 2c_p(T_c - T) = 2c_p T_c[1 - \frac{T}{T_c}]$$

So at any point in the nozzle where $s=s_c$

$$u = \sqrt{2c_p T_c}[1 - (\frac{p}{p_c})^{\frac{\gamma-1}{\gamma}}]^{\frac{1}{2}}$$

and this is independent of whether the nozzle is full at this pressure, because the area is not referred to. In particular, if we apply this at the end of the nozzle where the pressure is p_e, the velocity at that point is

$$u_e = \sqrt{2c_p T_c}[1 - (\frac{p_e}{p_c})^{\frac{\gamma-1}{\gamma}}]^{\frac{1}{2}}$$

You know from your previous work that <u>if the nozzle flows full</u> the thrust of the rocket can be written

$$F = \dot{m}u_e + A_e(p_e - p_o)$$

substituting the expression for u_e we then have the following expression for F in terms of the pressure ratio p_e/p_0.

$$F = \dot{m}\sqrt{2c_p T_c}[1 - (\frac{p_e}{p_c})^{\frac{\gamma-1}{\gamma}}]^{\frac{1}{2}} + A_e(p_e - p_o)$$

The mass flow rate is given in terms of c* by

$$\dot{m} \equiv \frac{p_c A_t}{c^*}$$

and for ideal gases, we have the estimate of c* given by

$$c^* = \frac{\sqrt{RT_c}}{\Gamma}; \quad \Gamma = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

F

Substituting these into the expression for F and non-dimensionalizing it:

$$\frac{F}{p_c A_t} = \sqrt{\frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \frac{A_e}{A_t}\left(\frac{p_e - p_o}{p_c}\right)$$

where the area ratio is itself related to the pressure ratio through continuity:

$$\frac{A_e}{A_t} = \frac{\rho_t u_t}{\rho_e u_e} = \sqrt{\frac{\gamma-1}{2}}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \frac{1}{\left(\frac{p_e}{p_c}\right)^{1/\gamma}\sqrt{1-\left(\frac{p_e}{p_c}\right)^{\frac{\gamma-1}{\gamma}}}}$$

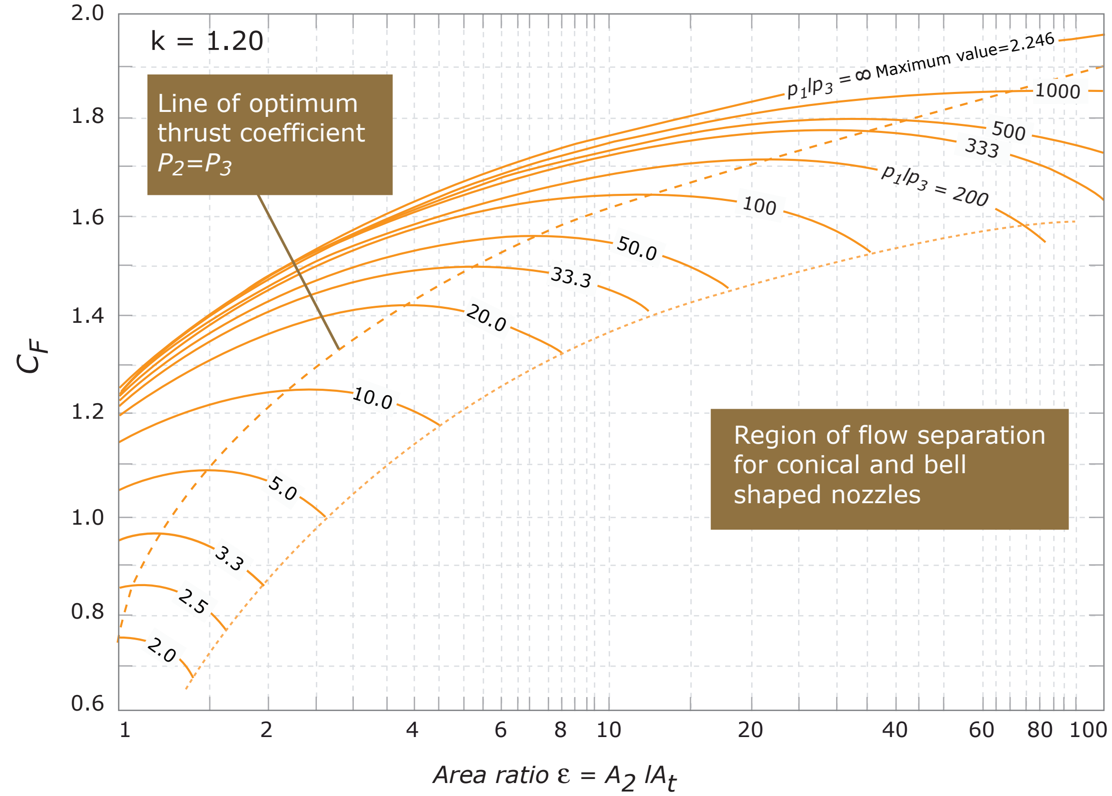
More generally, we define a Thrust Coefficient by

$$F \equiv p_c A_t c_F$$

and the above expression then gives us an estimate of c_F for ideal gases.

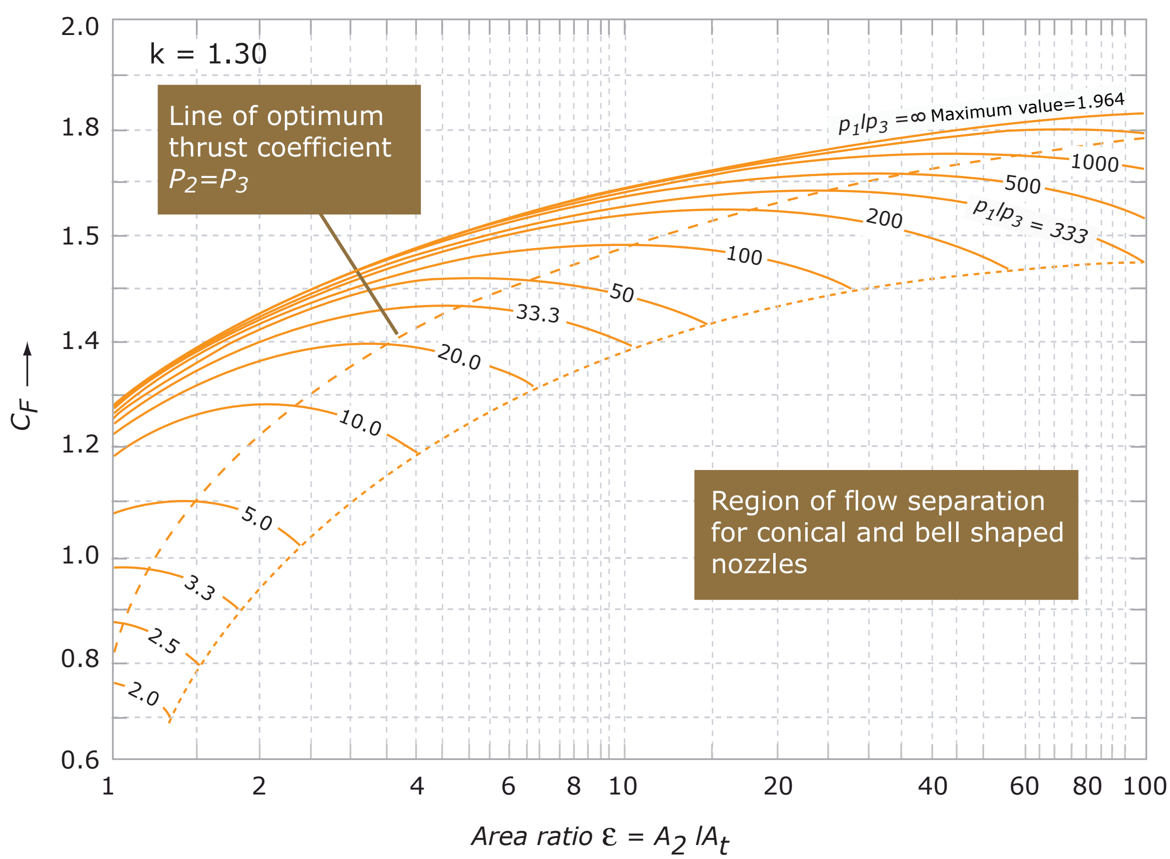
The dependence of c_F on nozzle area ratio and pressure ratio is conventionally summarized as in the figure below, which is drawn for γ=1.2 and 1.3. Notice:

-γ is replaced by k in these graphs
-p_c is replaced by p_1
-p_e is replaced by p_2
-p_a, or p_0, the ambient pressure, is replaced by p_3



Thrust coefficient C_F versus nozzle area ratio for k = 1.20.

Image by MIT OpenCourseWare. After Sutton and Biblarz 2001.



Thrust coefficient C_F versus nozzle area ratio for k = 1.30.

Image by MIT OpenCourseWare. After Sutton and Biblarz 2001.

The thrust is also expressible in terms of the effective exhaust velocity c, as we discussed in Lecture 1. Then we have

$$F \equiv \dot{m}c = \frac{p_c A_t}{c^*} = p_c A_t c_F$$

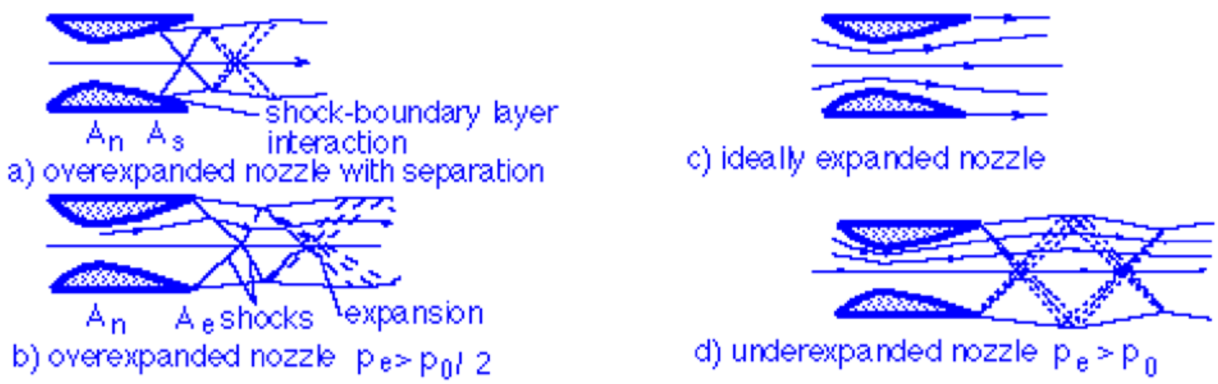
and we see that

$$c = c^* c_F$$

The "Characteristic Velocity" c* depends mainly on propellant properties. The Thrust Coefficient c_F depends on propellant properties through γ (or equivalent), but mainly on the pressure ratio and nozzle geometry. So by the definitions of c_F and c* we have managed to separate the effects of propellant properties and the effects of nozzle geometry into the two factors of c. This separation, though demonstrated only for ideal gases, holds also for the more general situation of complex chemically reacting propellants.

Effects of Non-ideal Expansion

So far we have assumed in the discussion of c_F that the flow fills the nozzle and is supersonic to the exit. If this is the case, the thrust is given by the above expression for c_F. But in fact the flow can be somewhat more complex than this, depending the ratio p_c/p_0 compared to that which leads to ideal expansion. This behavior is summarized in the figure below.



Here A_n is the throat area, A_e the exit area. The ideally expanded situation is at the upper right, and it is the condition at which thrust is maximum for a given external pressure p_0. This is easier to visualize than to prove analytically: if the divergent nozzle were extended a little by adding a section at its exit, this section would see an internal pressure lower than p_0, and would therefore generate suction, or negative thrust. If on the contrary, the nozzle were shortened a little, one would lose the positive thrust that was being produced by the removed portion.

Returning to consideration of a given nozzle, if p_0 is lowered below its ideal matching pressure (for example by the rocket ascending in the atmosphere) the nozzle becomes underexpanded, as at the lower right. In this case the flow fills the nozzle and our formula for F works fine.

If on the other hand p_0 is larger than corresponds to ideal expansion, the situation can become more complex. For pressure ratios $p_0/ p_e < 2$ to 2.5, the nozzle is likely to remain full, and again the formula holds. But for larger pressure ratios the oblique shocks that form at the exit of the nozzle are strong enough to separate the boundary layer, and the point of separation moves into the nozzle so that its effective area decreases, as shown at the upper left. In this case the separation occurs approximately at a pressure p_s such that $p_s/p_0 = ½$ to 1/(2.5).

Again we can use the formula for thrust by replacing p_e by p_s and A_e by A_s, the area at which the pressure is p_s. So now

$$F = \dot{m} u_s + A_s (p_s - p_o)$$

With this understanding, we can write the thrust coefficient for the separated nozzle as

$$\frac{F}{p_c A_t} = \sqrt{\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}} [1 - (\frac{p_s}{p_c})^{\frac{\gamma-1}{\gamma}}]} + \frac{A_s}{A_t}\left(\frac{p_s - p_o}{p_c}\right)$$

where now it is understood that p_s is the pressure at which the separation occurs and A_s is the corresponding area.

16.50 Introduction to Propulsion Systems
Spring 2012