16.50 Lecture 3

Subjects: Orbital mechanics; Single force center

The most usual application of rocket engines is to propel vehicles under conditions where the behavior of the vehicle is largely determined by the gravitational attractions of one or more bodies of the solar system, and where aerodynamic drag is not very important. So it is essential to understand the behavior of an orbiting body in order to appreciate the requirements which must be met by the rocket engine. For this reason we shall spend a short time discussing orbital mechanics. It should be noted, however, that the intent is to present only the aspects that define the requirements for propulsion systems, not to discuss the details of orbital computations. For these the student should refer to a text on celestial mechanics or spacecraft guidance.

Forces between bodies. The planetary Sphere of Influence

The gravitational attraction between two bodies of masses m and M is given by

$$Q = GmM\frac{1}{r^2}$$

where $G = 6.67\text{x}10^{-11}\text{m}^3\ \text{kg}^{-1}\ \text{s}^{-2}$, and r is the distance between the bodies. The gravitational acceleration is then $f = GM/r^2 \equiv \mu/r^2$, where μ=GM is the gravity constant of the body with mass M.

We are interested in the motion of a small body (spacecraft) in the force fields of one or more larger bodies. If we consider the Sun and two or more planets, the problem is extremely difficult mathematically. For our purposes, it can be simplified immensely by considering the motion of the spacecraft under the influence of only the dominant attractor at any given time. On the interplanetary scale, this means the Sun most of the time, but near enough one of the planets (inside its "Sphere of Influence", SOI), the planet will dominate. To estimate correctly the radius of the SOI, one should work in the frame of reference of the planet. The gravitational acceleration of the planet alone on the S/C is $f_{p,SC}$, and the perturbation due to the Sun of the spacecraft acceleration with respect to the planet is $f_{S,SC}-f_{S,p}$. Conversely, the gravitational acceleration of the Sun alone on the S/C is $f_{S,SC}$, and the perturbation due to the planet of the spacecraft acceleration with respect to the Sun is $f_{p,SC}-f_{p,S}$, which, since the distance to the Sun is much greater that that to the SC, is almost equal to $f_{p,SC}$ alone. We now state that the relative errors in ignoring either of the two perturbations are the same on the SOI:

$$\frac{f_{S,SC} - f_{S,p}}{f_{p,SC}} \simeq \frac{f_{p,SC}}{f_{S,SC}} \simeq \frac{f_{p,SC}}{f_{S,p}}$$

The difference on the left hand side numerator is approximated to first order as

$$f_{S,SC} - f_{S,p} \simeq \left|\frac{\partial f_S}{\partial r}\right| r_{p,SC} = 2\mu_S \frac{r_{p,SC}}{r_{S,p}^3}$$

Substituting this and the other accelerations into the SOI definition, and rearranging, one finds r_{SOI} (= $r_{p,SC}$) to be given by

$$\frac{r_{SOI}}{r_{S,p}} \simeq \frac{1}{2^{1/5}}\left(\frac{M_p}{M_S}\right)^{2/5}$$

Since M_p/M_S is a small number, $r_{SOI}/r_{S,p}<<1$. We can therefore model the relative body motion near the planets ($r<r_{SOI}$) as being under the influence of a single force center (the planet), while the body and the planet experience a common acceleration toward the Sun. Far from the planet ($r>r_{SOI}$) we ignore its field and consider the body motion as influenced only by the Sun. For preliminary calculations, these limiting models are simply "patched" at the SOI.

To see how far from a planet is far, we refer to Table 1-1, which gives the ratios of the planet's mass to that of the Sun and the size of hteir spheres of influence.

Table 1-1

	Relative mass	Orbital Radius, km	r_{SOI} (Km)
Mercury	$.167 \times 10^{-6}$	$.578 \times 10^{8}$	98,000
Venus	$.245 \times 10^{-5}$	1.08×10^{8}	536,000
Earth	$.300 \times 10^{-5}$	1.49×10^{8}	791,000
Mars	$.324 \times 10^{-6}$	2.27×10^{8}	501,000
Jupiter	$.956 \times 10^{-3}$	7.77×10^{8}	41.6×10^{6}
Saturn	$.286 \times 10^{-3}$	1.42×10^{9}	47.3×10^{6}
Uranus	$.437 \times 10^{-4}$	2.86×10^{9}	44.9×10^{6}
Neptune	$.518 \times 10^{-4}$	4.49×10^{9}	75.5×10^{6}
Pluto	$.28 \times 10^{-5}$	5.89×10^{9}	30.8×10^{6}
Earth's Moon,	$M_M/M_E=.368 \times 10^{-7}$	$r_{E,M}=.384 \times 10^{6}$	57,600

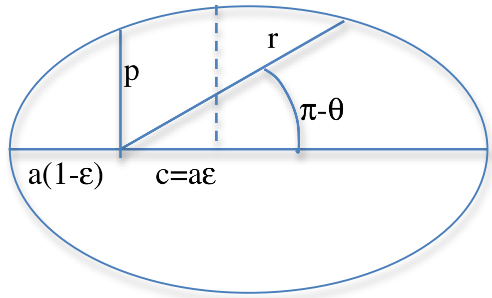
We see for example that the sphere of influence of the Earth is just about twice the distance to the Moon. On one hand this means we can just barely ignore solar perturbations in considering cis-lunar operations not very near the Moon, but also that the Moon's own motion about the Earth must be appreciably affected by the Sun's attraction. Notice also the relatively large size of the SOI of the Moon in the Earth-Moon system (15% of the Earth/Moon distance).

We can therefore reduce the complex many-force center situation to a set of simpler ones, namely

a) Motion in the single force field of a <u>planet</u>
b) Motion in the single force field of the Sun
c) Transition from a Sun-dominated to a planet dominated situation, or vice versa. As we shall see this transition can be thought of as a <u>change of coordinate system.</u>

Motion under a Single Force Center

The subject of motion under a single gravitational attractor is covered in all Dynamics textbooks, and so we only give here a summary of the main conclusions and working equations.



With reference to the figure, the polar equation of the trajectory is $r = \frac{p}{1+\varepsilon\cos\theta}$,

where $p = a(1-\varepsilon^2)$, ε is the eccentricity and a is the semi-major axis. If ε<1, all the quantities listed are positive and the trajectory is closed (an ellipse with a focus at r=0). The case when ε>0 will be discussed later. The quantity p is called "the parameter", and has the significance shown in the figure. The minimum and maximum radii are called the "periapsis" and "apoaxis", respectively (perigee and apogee for orbits around Earth); they are given by

$$r_p = \frac{p}{1+\varepsilon} = a(1-\varepsilon); \quad r_a = \frac{p}{1-\varepsilon} = a(1+\varepsilon)$$

The potential energy per unit mass, with zero at infinity, is $E_p = -\mu / r$.

The two constants of the motion are:

(a) The total energy per unit mass, $E = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$

(b) The angular momentum per unit mass, $h = r^2\dot{\theta} = \sqrt{\mu p} = \sqrt{a(1-\varepsilon^2)}$

The time to complete one orbit is $T = 2\pi \frac{a^{3/2}}{\sqrt{\mu}}$

Using the conservation of energy, the velocity magnitude is given by the so-called "vis-viva" equation: $v = \sqrt{\frac{2\mu}{r} - \frac{\mu}{a}}$

The circular orbital velocity (r=a) is therefore $v_c = \sqrt{\frac{\mu}{r}} = \sqrt{\frac{\mu}{a}}$

Notice that the kinetic energy in a circular orbit is $E_K = \mu / 2a = -E = -1/2\, E_p$. More generally, the average kinetic and potential energies satisfy these same curious relationships for any elliptic orbit.

The trajectory first becomes open when ε=1, r_a tends to infinity, and the total energy becomes zero. When this happens, the body is on a trajectory to barely "escape" the attractor, arriving at infinity with zero velocity. The velocity on this trajectory depends on distance as $v_{esc} = \sqrt{\frac{2\mu}{r}}$.

As a final note, the apogee and perigee velocities occur frequently in orbital calculations. They are related to the apogee and perigee radii by

$$v_p = \sqrt{\frac{\mu}{r_p}\frac{2r_a}{r_a + r_p}} > v_{c,p}; \quad v_a = \sqrt{\frac{\mu}{r_a}\frac{2r_p}{r_a + r_p}} < v_{c,a}$$

Although the formulation in terms of the classical “orbital elements” a, ε, p, etc, is standard, most simple problems can be solved quickly by using the energy and angular momentum conservation laws between judiciously selected points, often the apogee and perigee themselves.