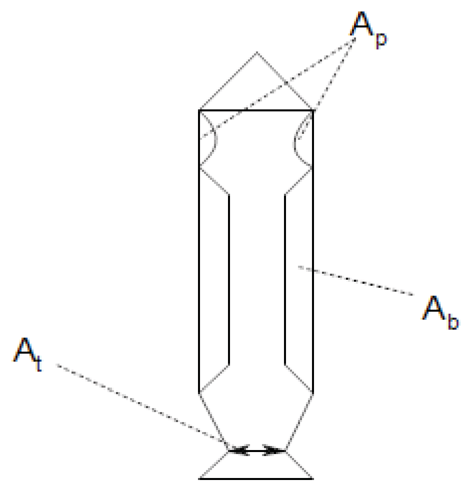
Homework 4.1: Solid Propellant Rocket



1a) Normal operation mass balance:

$a\rho_p A_b P_{c0}^n - \frac{A_t P_{c0}}{c^*} = 0$ (1)

Transient operation after port opening:

$\frac{V_c}{RT_c}\frac{dP_c}{dt} = a\rho_P A_b P_c^n - \frac{(A_t+A_b)P_c}{c^*}$ (2)

Define:

$t_{ch} = \frac{V_c c^*}{RT_c A_b}$ (3)

$\tau = \frac{t}{t_c}$ (4)

$\alpha = \frac{A_P}{A_t}$ (5)

$y = \frac{P_c}{P_{c0}}$ (6)

In terms of these variables, equation (2) becomes:

$\frac{dy}{d\tau} = y^n - (1+\alpha)y$ (7)

Multiply equation (7) by y^{-n}:

$y^{-n}\frac{dy}{d\tau} = 1 - (1+\alpha)y^{1-n}$ (8)

$(1-n)y^{-n}dy = d(y^{1-n})$

Define:

$y^{1-n} = u$ (9)

$\frac{1}{1-n}\frac{du}{d\tau} = 1 - (1+\alpha)u$ (10)

This relationship is *linear* and can be easily solved. The *particular solution* is $u = \frac{1}{1+\alpha}$, and the *homogeneous solution* is $u = e^{-(1-n)(1+\alpha)\tau}$. The *complete solution* is then $u = \frac{1}{1+\alpha} + Ae^{-(1-n)(1+\alpha)\tau}$, where A is arbitrary.

At $t = 0$ ($\tau = 0$), we have $P_c = P_{c0}$ ($y = u = 1$), so

$1 = \frac{1}{1+\alpha} + A$

$A = \frac{\alpha}{1+\alpha}$

Therefore:

$u = \frac{1+\alpha e^{-(1-n)(1+\alpha)\tau}}{1+\alpha}$

Using $y = u^{\frac{1}{1-n}}$:

$$y = \left[\frac{1+\alpha e^{-(1-n)(1+\alpha)\tau}}{1+\alpha}\right]^{\frac{1}{1-n}} \qquad (11)$$

1b) The combustion stops when $P_c \le 20\ atm$ $\left(y_{extinction} = \frac{20}{70} = \frac{2}{7}\right)$. For $t \to \infty$ $(\tau \to \infty)$, we obtain from equation (11):

$$y(\infty) = \frac{1}{(1+\alpha)^{\frac{1}{1-n}}} \qquad (12)$$

For this to be equal or less than $\frac{2}{7}$, α must be more than:

$$\alpha_{min} = \left(\frac{1}{y(\infty)}\right)^{1-n} - 1 = \left(\frac{7}{2}\right)^{1-0.2} - 1 \qquad (13)$$

$$\left(\frac{A_P}{A_t}\right)_{min} = 1.724 \qquad (14)$$

For values of $\alpha > 1.724$, the extinction limit $y = \frac{2}{7}$ is reached in a finite time. Solving **equation (11)** for τ gives:

$$\tau_{ext} = \frac{1}{(1+\alpha)(1-n)}\ ln\left(\frac{\alpha}{(1+\alpha)y_{ext}^{1-n}-1}\right) \qquad (15)$$

Since $n = 0.2$ and $(1+\alpha)y_{ext}^{1-n} = \frac{(1+\alpha)}{(1+\alpha_{\min})}(1+\alpha_{\min})y_{ext}^{1-n}$:

$$\tau_{ext} = \frac{1}{0.8(1+\alpha)} ln\left(\frac{\alpha}{\frac{1+\alpha}{2.724}-1}\right) \qquad (16)$$

$$t_{ext} = \tau_{ext}\frac{V_c c^*}{RT_c A_t} \qquad (17)$$

Assuming a molecular mass $M = 20\frac{g}{mole}$ (not specified in problem statement),

$$\frac{V_c c^*}{RT_c A_t} = \frac{10\times1800}{\frac{8.314}{0.02}\times3400} = 1.274\times10^{-2}\ s \qquad (18)$$

We can now calculate a few extinction times corresponding to choices of $\frac{A_P}{A_t}$ above the minimum, shown in **Table 1**.

Table 1: Extinction Times

$\alpha = \frac{A_P}{A_t}$	1.724	2	3	4
τ_{ext}	∞	1.243	0.5803	0.3915
$t_{ext}\ [s]$	∞	1.583e-2	7.42e-3	4.99e-3

Homework 4.2: Monopropellant Hydrazine Rocket

Rapid disproportionation: $N_2H_4 \rightarrow \frac{4}{3}NH_3 + \frac{1}{3}N_2$ (1)

Slow NH_3 decomposition: $\frac{4}{3}NH_3 \rightarrow \frac{2}{3}N_2 + 2H_2$ (2)

Since we assume 40% Ammonia decomposition, form **equation (1)** + 0.4***equation (2):**

$$N_2H_4 \rightarrow \frac{4}{3}(1-0.4)NH_3 + \frac{1+0.8}{3}N_2 + 0.4*2H_2 \quad (3)$$

$$N_2H_4 \rightarrow 0.8NH_3 + 0.6N_2 + 0.8H_2 \quad (4)$$

We can now write the enthalpy balance for the reaction. The enthalpy of the reactants (liquid Hydrazine at 298.2K) is +50.63 kJ/mol, so using the fits provided for gaseous NH_3, N_2, and H_2:

$$50.63 = 0.8(-70.40 + 51.166\theta + 4.11\theta^2) + 0.6(-11.84 + 32.42\theta + 0.76\theta^2) + \cdots$$
$$+0.8(-8.23 + 27.16\theta + 1.34\theta^2)$$
$$4.976\theta^2 + 82.508\theta - 120.64 = 0$$

$$\theta = \frac{-82.508+\sqrt{82.508^2+4*4.976*120.64}}{2*4.976} = 1.352 = \frac{T}{1000} \quad (5)$$
$$T = 1{,}352K$$

Mean molecular mass:

$$\bar{M} = \frac{0.8*17+0.6*28+0.8*2}{0.8+0.6+0.8} = 16.0\frac{g}{mol} = 0.016\frac{kg}{mol}$$

Mean specific heat:

$$c_p = \frac{\left[0.8(\widetilde{c_p})_{NH_3} + 0.6(\widetilde{c_p})_{N_2} + 0.8(\widetilde{c_p})_{H_2}\right] \Big/ (0.8+0.6+0.8)}{0.016}$$

$$(\widetilde{c_p})_{NH_3} = \frac{\partial h_{NH_3}}{\partial T} = \frac{1}{1000}\frac{\partial h_{NH_3}}{\partial \theta} = \frac{51.66}{1000}\frac{kJ}{mol*K} = 51.66\frac{J}{mol*K}$$
$$(\widetilde{c_p})_{N_2} = \frac{\partial h_{N_2}}{\partial T} = \frac{1}{1000}\frac{\partial h_{N_2}}{\partial \theta} = \frac{32.42}{1000}\frac{kJ}{mol*K} = 32.42\frac{J}{mol*K}$$
$$(\widetilde{c_p})_{H_2} = \frac{\partial h_{H_2}}{\partial T} = \frac{1}{1000}\frac{\partial h_{H_2}}{\partial \theta} = \frac{27.61}{1000}\frac{kJ}{mol*K} = 27.61\frac{J}{mol*K}$$

$$c_p = \frac{0.4*51.66+0.3*32.42+0.4*27.61}{0.016} = 2{,}590\frac{J}{kg*K}$$

We could now calculate $c_v = 2{,}590 - \frac{8.314}{0.016} = 2.070\frac{J}{kg*K}$ and so $\gamma = \frac{c_p}{c_v} = 1.2512$

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