Homework 3: Two-Position Nozzle

a) $c^* = \frac{\sqrt{RT_c}}{\Gamma(\gamma)}$ (1)

$\Gamma = \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$ (2)

$\Gamma = \sqrt{1.2}\left(\frac{2}{2.2}\right)^{\frac{2.2}{2*0,2}} = 0.6485$

$R = \frac{8.314\ ^{J}/_{mol-K}}{0.018\ ^{kg}/_{mol}} = 461.9\ ^{J}/_{mol-K}$ (3)

Solving for c^* and $\dot{m}$:

$c^* = \frac{\sqrt{461.9*3300}}{0.6485} = 1903.7\ m/s$

$\dot{m} = \frac{P_c A_t}{c^*} = \frac{70*1.013E5*0.1}{1903.7} = 372.5\ kg/s$

For the inner bell:

$\frac{P_{e1}}{P_{a0}} = 0.4$

Therefore, the exit Mach number is given by:

$\frac{70}{0.4} = \left(1 + \frac{1.2-1}{2} M_{e1}^2\right)^{\frac{1.2}{0.2}}$

$M_{e1} = 3.695$

The area ratio is then:

$\frac{A_{e1}}{A_t} = \frac{1}{M_{e1}}\left(\frac{1+\frac{\gamma-1}{2}M_{e1}^2}{\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$ (4)

$\frac{A_{e1}}{A_t} = 18.234$

$A_{e1} = 1.8234\ m^2$

The vacuum thrust is:

$F_{v1} = \dot{m} u_{e1} + P_{e1} A_{e1}$ (5)

$u_{e1} = \sqrt{2\frac{\gamma}{\gamma-1} RT_c\left[1 - \left(\frac{P_{e1}}{P_c}\right)^{\frac{\gamma-1}{\gamma}}\right]}$ (6)

$u_{e1} = \sqrt{2\frac{1.2}{0.2}461.9 * 3300\left[1 - \left(\frac{0.4}{70}\right)^{\frac{0.2}{1.2}}\right]} = 3241\ m/s$

On the ground:

$F_0 = F_{v1} - P_{a0}A_{e1}$ (7)

$F_0 = 1.2842E6 - 1.013E5 * 1.8234 = 1.0995E6\ N$

$F_{v1} = \dot{m}u_{e1} + P_{e1}A_{w1} = 372.5 \times 3421 + 0.4 \times 1.013e5 \times 1.8234 = 1.2842e6\ N$

b) $m\frac{dv}{dt} = F - mg = F_v - P_aA_e - mg$ (8)

$\frac{dv}{dt} = \frac{F_v}{m} - g - \frac{A_eP_a}{m}$ (9)

$m = m_0 - \dot{m}t$ (10)

$dm = -\dot{m}dt$ (11)

Putting equations together:

$dv = -\frac{F_v}{m}\frac{dm}{m} - gdt - \frac{A_eP_a}{m}dt$ (12)

Integrating between $t = 0$ and $t = t_1$ (with $F_v = F_{v1}$ and $A_e = A_{e1}$) :

$v_1 = \frac{F_{v1}}{\dot{m}} ln\left(\frac{m_0}{m_1}\right) - gt_1 - A_{e1}\int_0^{t_1}\frac{P_a(t)}{m(t)}dt$ (13)

Integrating between $t = t_1$ and $t = t_b$ (with $F_v = F_{v1}$ and $A_e = A_{e2}$):

$v_b - v_1 = \frac{F_{v2}}{\dot{m}} ln\left(\frac{m_1}{m_b}\right) - g(t_b - t_1) - A_{e2}\int_{t_1}^{t_b}\frac{P_a(t)}{m(t)}dt$ (14)

Adding equations (13) and (14):

$v_b = \frac{F_{v1}}{\dot{m}} ln\left(\frac{m_0}{m_1}\right) + \frac{F_{v2}}{\dot{m}} ln\left(\frac{m_1}{m_b}\right) - gt_b - A_{e1}\int_0^{t_1}\frac{P_a(t)}{m(t)}dt - A_{e2}\int_{t_1}^{t_b}\frac{P_a(t)}{m(t)}dt$ (15)

Here, of course:

$m_1 = m_0 - \dot{m}t_1$ (16)

So t_1 appears in m_1 and in the limits of integration.

To optimize, set $\frac{dv_b}{dt_1} = 0$, using $\frac{dm_1}{dt_1} = -\dot{m}$

$\frac{-F_{v1}}{m}\frac{1}{m_1}(-m) + \frac{F_{v2}}{m}\frac{1}{m_1}(-m) - A_{e1}\frac{P_a(t_1)}{m_1} + A_{e2}\frac{P_a(t_1)}{m_1} = 0$

$F_{v1} - F_{v2} + (A_{e2} - A_{e1})P_a(t_1) = 0$

$P_a(t_1) = \frac{F_{v2}-F_{v1}}{A_{e2}-A_{e1}}$ (for optimum t_1)

If this t_1 is chosen, the thrust is:

Just before the transition:

$F_1(t_1 - \varepsilon) = F_{v1} - \frac{F_{v2}-F_{v1}}{A_{e2}-A_{e1}}A_{e1} = \frac{F_{v1}A_{e2}-F_{v2}A_{e1}}{A_{e2}-A_{e1}}$ (17)

Just after the transition:

$$F_2(t_1+\varepsilon) = F_{v2} - \frac{F_{v2}-F_{v1}}{A_{e2}-A_{e1}} A_{e2} = \frac{F_{v1}A_{e2}-F_{v2}A_{e1}}{A_{e2}-A_{e1}} \qquad (18)$$

Therefore,
$F_1(t_1-\varepsilon) = F_2(t_1+\varepsilon)$ There is no discontinuity in the thrust.

c) If we impose now $P_{e2} = 0.4P_a(t_1)$ (incipient separation on the extended nozzle), we must have:

$$P_{e2} = 0.4\frac{F_{v2}-F_{v1}}{A_{e2}-A_{e1}} = 0.4\frac{(\dot{m}u_{e2}+P_{e2}A_{e2})-F_{v1}}{A_{e2}-A_{e1}} \qquad (19)$$

Here, $\dot{m}$, F_{v1}, and A_{e1} are already known (from part a), and all the other quantities (u_{e2}, P_{e2}, A_{e2}) depend on a single parameter like M_{e2}. Hence the equation above determines M_{e2}, and all other quantities. Equation (19) can be solved by trial and error as follows:

a) Guess M_{e2}

b) Calculate $A_{e2} = \frac{A_t}{M_{e2}}\left(\frac{1+\frac{\gamma-1}{2}M_{e2}^2}{\frac{\gamma+1}{2}}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$

c) Calculate $T_{e2} = \frac{T_c}{1+\frac{\gamma-1}{2}M_e^2}$, then $u_{e2} = M_{e2}\sqrt{\gamma R T_{e2}}$

d) Calculate $P_{e2} = P_c\left(\frac{T_{e2}}{T_c}\right)^{\frac{\gamma}{\gamma-1}}$

e) Calculate the right hand side of equation (19), compare to P_{e2} from the left hand side.

Results of this process are plotted in Figures 1a and 1b, and the solution is seen to be $M_{e2} = 4.815$. From this, we find:

$\frac{A_{e2}}{A_t} = 90.13$

$A_{e2} = 9.013\ m^2$

$u_{e2} = 3{,}575\ m/s$

$P_{e2} = 0.05236\ atm$

$F_{v2} = \dot{m}u_{e2} + P_{e2}A_{e2} = 1.3795E6\ N$

$P_a(t_1) = \frac{F_{v2}-F_{v1}}{A_{e2}-A_{e1}} = 0.1308\ atm$

$z_1 = 6.8 ln\left(\frac{1}{0.1308}\right) = 13.83\ km$

The **thrust at transition** (from either nozzle 1 or 2) is:
$F(t_1) = F_{v1} - P_a(t_1)A_{e1} = 1.2570E6\ N$

c) For the ideally expanded ("rubber") nozzle, $P_e = P_a(z)$, and then:

$$u_e = \sqrt{2\frac{\gamma}{\gamma-1}RT_c\left[1-\left(\frac{P_a(z)}{P_c}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

$F = \dot{m}u_e$

For $z = 0, z = z_1$, and $z \to \infty$ $(P_a = 0)$, and for the three types of nozzles, Table 1 collects the values of thrust:

Table 1: Thurst calculations for three nozzles at varying altitudes.

Thrust (MN)			
Nozzle	$z = 0$	$z = z_1 = 13.83\ km$	$z \to \infty$
Fixed Geometry	1.0995	1.2570	1.2842
Two-Position	1.0995	1.2570	1.3795
Ideally Expanded	1.1348	1.2836	1.5931

Notice:

a) Thrust increases with altitude in all cases.

b) The two-position nozzle is equivalent to the fixed nozzle at z_1, but clearly superior at the vacuum condition.

c) The ideally expanded nozzle outperforms the others at all altitudes.

Figure 2 compares the thrust profiles of the thrust nozzles. The two-position nozzle would follow the curve for nozzle 1 up to $z = z_1$, then follow the nozzle 2 curve.

Notice how the ideally expanded curve touches that for nozzle 1 at $z \approx 6.23\ km$; that is where this nozzle is matched $(6.8ln\left(\frac{1}{0.4}\right) = 6.23)$. It also touches that of nozzle 2 at $z = 6.8ln\left(\frac{1}{0.05236}\right) = 20.1\ km$, where nozzle 2 is matched.

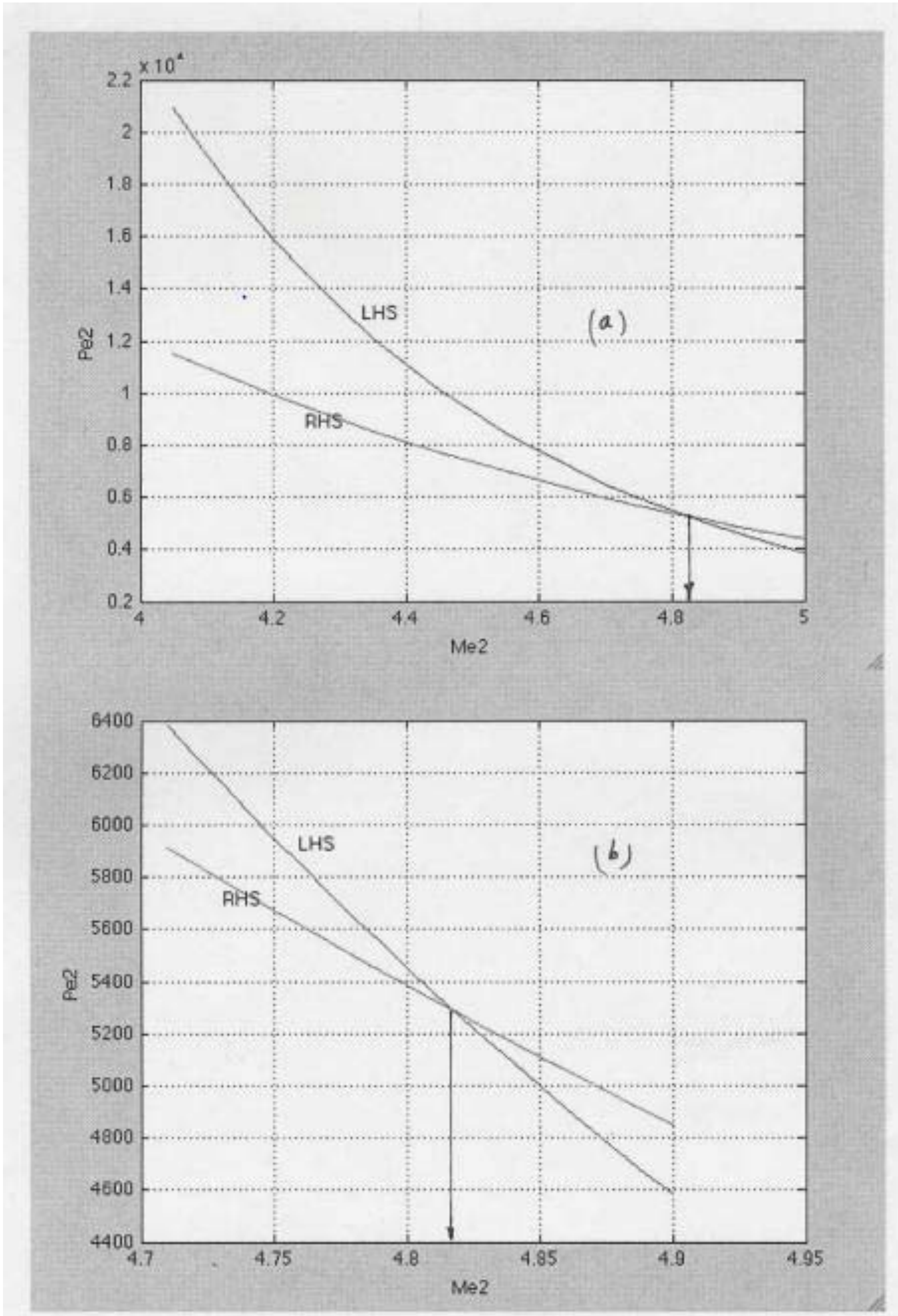

Figure 1a) Solving equation (19).

Figure 1b) Solving equation (19) at a higher resolution. Solution is $M_{e2} = 4.815$.

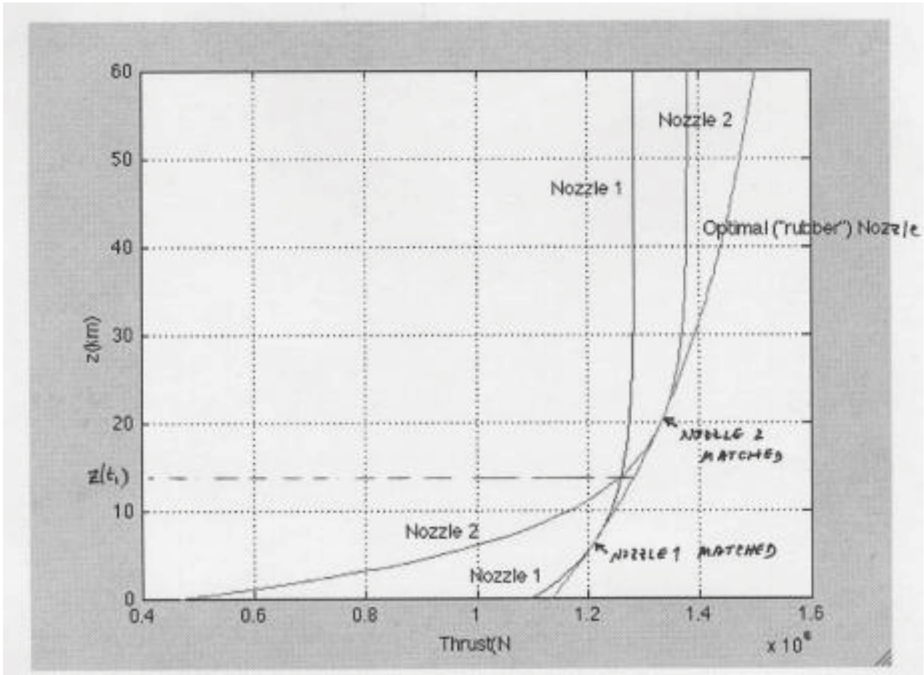

Figure 2: Thrust profiles for the three nozzles.

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