16.36 Communication Systems Engineering
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Lectures 11: Hypothesis Testing and BER analysis

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Signal Detection

- After matched filtering we receive $r = S_m + n$
  - $S_m \in \{S_1,..S_M\}$
- How do we determine from r which of the M possible symbols was sent?
  - Without the noise we would receive what sent, but the noise can transform one symbol into another

<u>Hypothesis testing</u>

- Objective: minimize the probability of a decision error
- Decision rule:
  - Choose S_m such that P(S_m sent | r received) is maximized
- This is known as Maximum a posteriori probability (MAP) rule
- MAP Rule: Maximize the conditional probability that S_m was sent given that r was received

MAP detector

MAP detector: $\max_{S_1 \ldots S_M} P(S_m \mid r)$

$$P(S_m \mid r) = \frac{P(S_m, r)}{P(r)} = \frac{P(r \mid S_m)P(S_m)}{P(r)}$$

$$P(S_m \mid r) = \frac{f_{r|s}(r \mid S_m)P(S_m)}{f_r(r)}$$

$$f_r(r) = \sum_{m=1}^{M} f_{r|s}(r \mid S_m)P(S_m)$$

When $P(S_m) = \frac{1}{M}$ Map rule becomes:

$\max_{S_1 \ldots S_M} f(r \mid S_m)$ (AKA Maximum Likelihood (ML) decision rule)

- **Notes:**
  - **MAP rule requires prior probabilities**
  - **MAP minimizes the probability of a decision error**
  - **ML rule assumes equally likely symbols**
  - **With equally likely symbols MAP and ML are the same**

Detection in AWGN
(Single dimensional constellations)

$$f(r \mid S_m) = \frac{1}{\sqrt{\pi N_0}} e^{-(r-S_m)^2 / N_0}$$

$$\ln(f(r \mid S_m)) = -\ln(\sqrt{\pi N_0}) - \frac{(r-S_m)^2}{N_0}$$

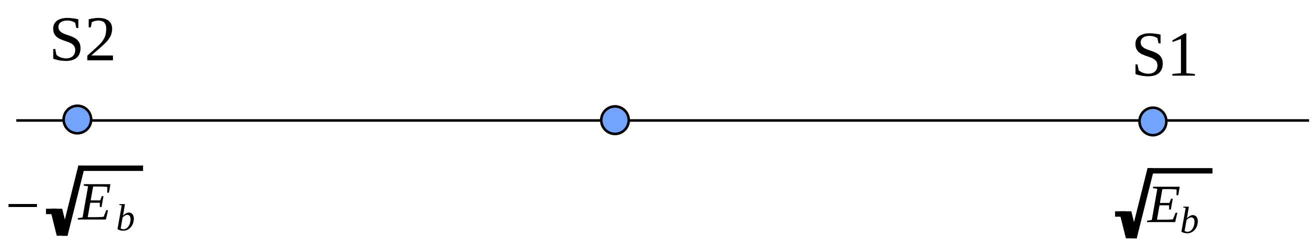
$$d_{rS_m} = (r-S_m)^2$$

Maximum Likelihood decoding amounts to minimizing $d_{rS_m} = (r-S_m)^2$

- **Also known as minimum distance decoding**
  - **Similar expression for multidimensional constellations**

Detection of binary PAM

- **S1(t) = g(t), S2(t) = -g(t)**
  - **S1 = - S2 => “antipodal” signaling**
- **Antipodal signals with energy Eb can be represented geometrically as**

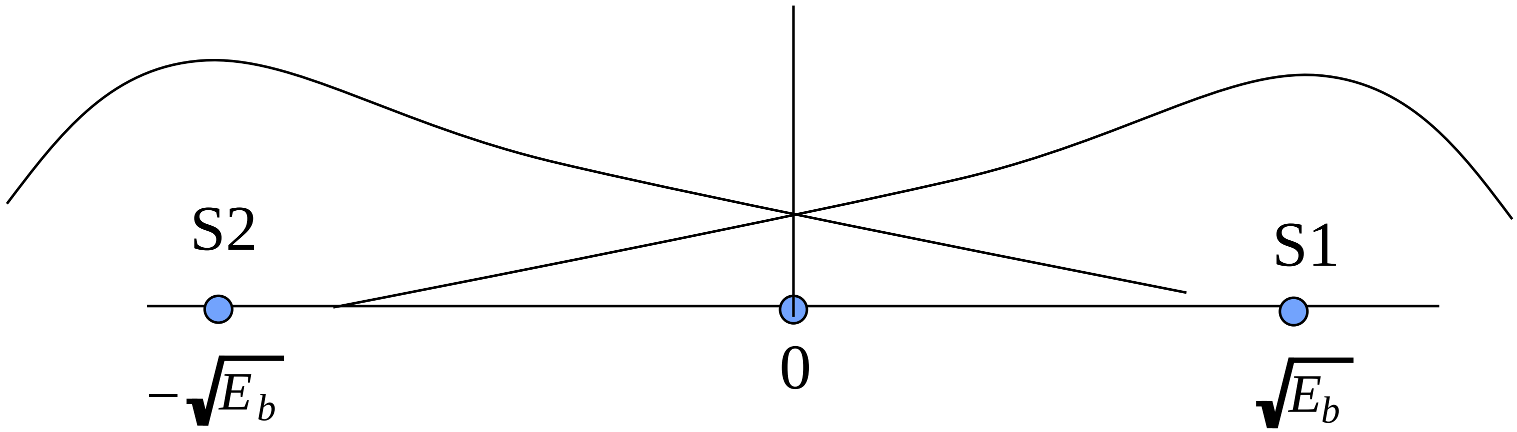


- **If S1 was sent then the received signal r = S1 + n**
- **If S2 was sent then the received signal r = S2 + n**

$$f_{r|s}(r \mid s1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{E_b})^2 / N_0}$$

$$f_{r|s}(r \mid s2) = \frac{1}{\sqrt{\pi N_0}} e^{-(r+\sqrt{E_b})^2 / N_0}$$

Detection of Binary PAM



- **Decision rule: MLE => minimum distance decoding**
  - **=> r > 0 decide S1**
  - **=> r < 0 decide S2**

- **Probability of error**
  - **When S2 was sent the probability of error is the probability that noise exceeds $(Eb)^{1/2}$ similarly when S1 was sent the probability of error is the probability that noise exceeds - $(Eb)^{1/2}$**
  - **P(e|S1) = P(e|S2) = P[r<0|S1)**

Probability of error for binary PAM

$$P_e = \int_{-\infty}^{0} f_{r|s}(r \mid s1)dr = \int_{-\infty}^{0} \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{E_b})^2/N_0} dr$$

$$= \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{-\sqrt{E_b}} e^{-r^2/N_0} dr$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{2E_b/N_0}} e^{-r^2/2} dr$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2E_b/N_0}}^{\infty} e^{-r^2/2} dr$$

$$\equiv Q(\sqrt{2E_b/N_0}) \quad where,$$

$$Q(x) \,\underline{\Delta}\, \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-r^2/2} dr$$

- Q(x) = P(X>x) for X Gaussian with zero mean and $\sigma^2 = 1$
- Q(x) requires numerical evaluation and is tabulated in many math books (Table 4.1 of text)

More on Q function

- **Notes on Q(x)**
  - **Q(0) = 1/2**
  - **Q(-x) = 1-Q(x)**
  - **Q(∞) = 0, Q(- ∞)=1**

  - **If X is N(m,σ^2) Then P(X>x) = Q((x-m)/ σ)**

- **Example: Pe = P[r<0|S1 was sent)**

$$f_{r|s}(r \mid s1) \sim N(\sqrt{E_b}, N_0/2) \Rightarrow m = \sqrt{E_b}, \sigma = \sqrt{N_0/2}$$

$$P_e = 1 - P[r > 0 \mid s1] = 1 - Q\left(\frac{-\sqrt{E_b}}{\sqrt{N_0/2}}\right) = 1 - Q(-\sqrt{2E_b/N_0}) = Q(\sqrt{2E_b/N_0})$$

Error analysis continued

- **In general, the probability of error between two symbols separated by a distance d is given by:**

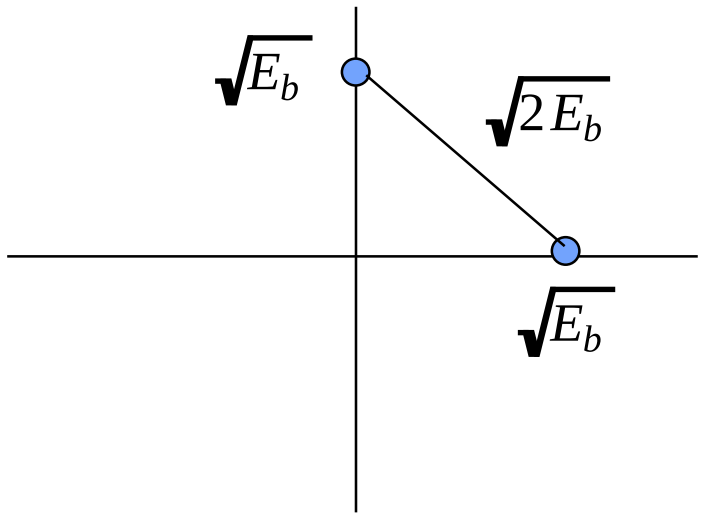
$$P_e(d) = Q\left(\sqrt{\frac{d^2}{2N_0}}\right)$$

- **For binary PAM d = 2** $\sqrt{E_b}$ **Hence,**

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Orthogonal signals

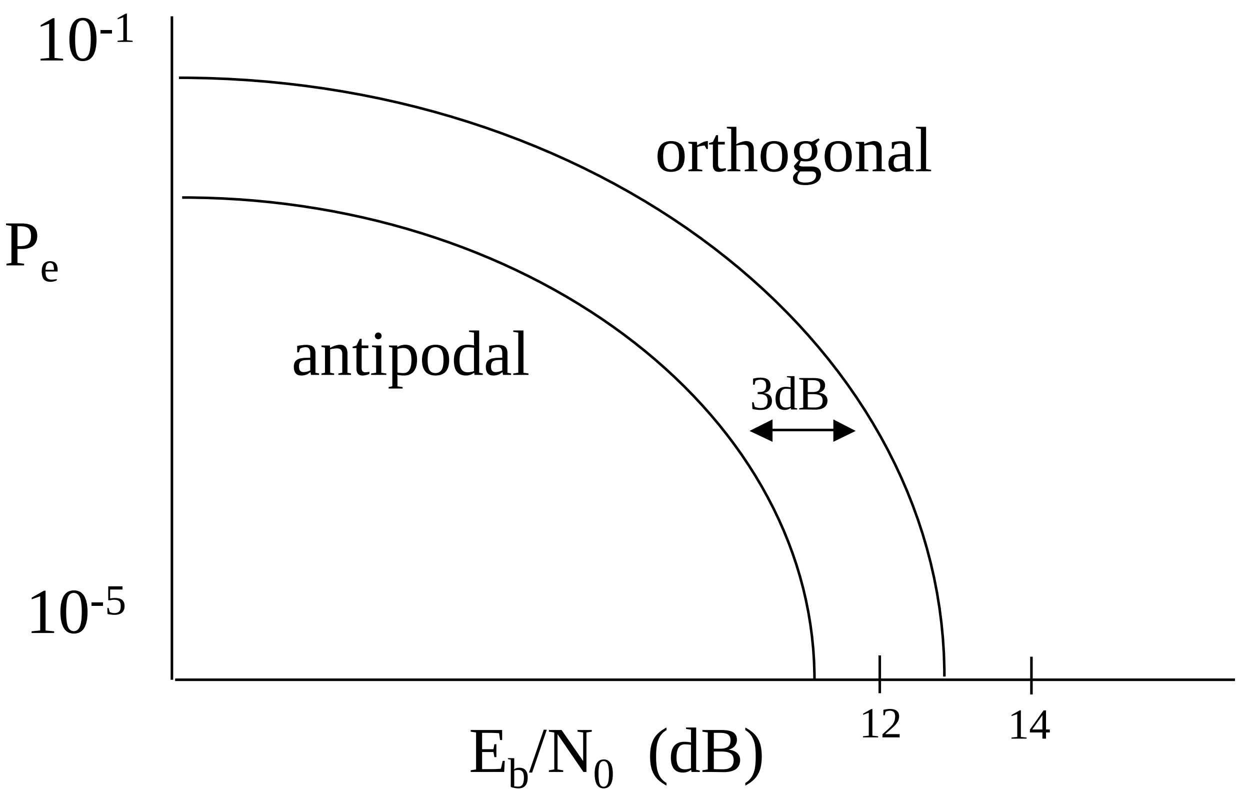
- **Orthogonal signaling scheme (2 dimensional)**



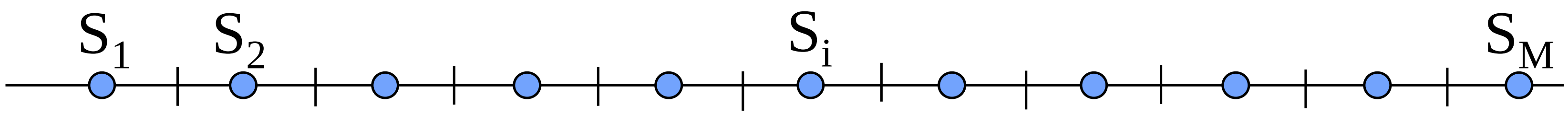
$$P_e = Q(\sqrt{\frac{d^2}{2N_0}} = Q(\sqrt{E_b / N_o})$$

Orthogonal vs. Antipodal signals

- **Notice from Q function that orthogonal signaling requires twice as much bit energy than antipodal for the same error rate**
  - **This is due to the distance between signal points**



Probability of error for M-PAM



$$S_M = A_M \sqrt{E_g}, \quad A_M = (2m - 1 - M)$$

$$d_{ij} = 2\sqrt{E_g} \quad for\ |i - j| = 1$$

Decision rule: Choose s_i such that $d(r, s_i)$ is minimized

$$P[\text{error} \mid s_i] = P[decode\ \ s_{i-1} \mid s_i] + P[decode\ \ s_{i+1} \mid s_i] = 2P[decode\ \ s_{i+1} \mid s_i]$$

$$Pe = 2Q\left[\sqrt{\frac{d_{i,i+1}^2}{2N_0}}\right] = 2Q\left[\sqrt{\frac{2E_g}{N_0}}\right], \quad P_{eb} = \frac{Pe}{Log_2(M)}$$

<u>Notes:</u>

1) the probability of error for s_1 and s_M is lower because error only occur in one direction
2) With Gray coding the bit error rate is $P_e/\log_2(M)$

Probability of error for M-PAM

$$E_{av} = \frac{M^2 - 1}{3} E_g \Rightarrow E_{bav} = \frac{M^2 - 1}{3 Log_2(M)} E_g$$

$$E_g = \frac{3 Log_2(M)}{M^2 - 1} E_{bav}$$

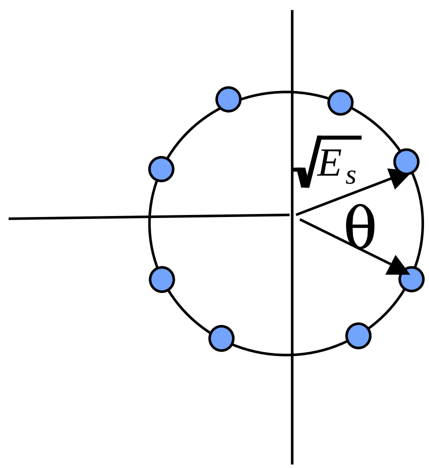
$$P_e = 2Q\left[\sqrt{\frac{6 Log_2(M)}{(M^2-1)N_0} E_{bav}}\right], \quad P_{eb} = \frac{Pe}{Log_2(M)}$$

accounting for effect of S_1 *and* S_M *we get:*

$$P_e = 2\left(\frac{M-1}{M}\right)Q\left[\sqrt{\frac{6 Log_2(M)}{(M^2-1)N_0} E_{bav}}\right],$$

Probability of error for PSK

- Binary PSK is exactly the same as binary PAM
- 4-PSK can be viewed as two sets of binary PAM signals
- For large M (e.g., M>8) a good approximation assumes that errors occur between adjacent signal points



$$\theta = 2\pi/M$$

$$d_{ij} = 2\sqrt{E_s}Sin(\frac{\pi}{M}), \qquad |i-j| = 1$$

Error Probability for PSK

$$\text{P[error}|\text{s}_\text{i}] = P[decode\ \ s_{i-1} \mid s_i] + P[decode\ \ s_{i+1} \mid s_i] = 2P[decode\ \ s_{i+1} \mid s_i]$$

$$P_{es} = 2Q\left[\sqrt{\frac{d_{i,i+1}^2}{2N_0}}\right] = 2Q\left[\sqrt{\frac{2E_s}{N_0}}\sin(\pi / M)\right]$$

$$E_b = E_s / Log_2(M)$$

$$P_{es} = 2Q\left[\sqrt{\frac{2Log_2(M)E_b}{N_0}}\sin(\pi / M)\right], \quad P_{eb} = \frac{P_{es}}{Log_2(M)}$$