16.36 Communication Systems Engineering
Spring 2009

Lectures 8 - 9: Signal Detection in Noise and the Matched Filter

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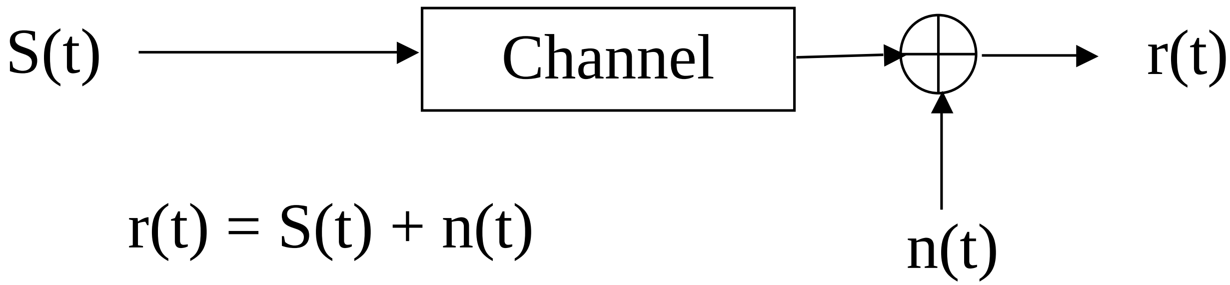
Noise in communication systems



$$r(t) = S(t) + n(t)$$

- **Noise is additional "unwanted" signal that interferes with the transmitted signal**
  - **Generated by electronic devices**
- **The noise is a random process**
  - **Each "sample" of n(t) is a random variable**
- **Typically, the noise process is modeled as "Additive White Gaussian Noise" (AWGN)**
  - **White: Flat frequency spectrum**
  - **Gaussian: noise distribution**

Random Processes

- The auto-correlation of a random process x(t) is defined as
  - $R_{xx}(t_1,t_2) = E[x(t_1)x(t_2)]$

- A random process is Wide-sense-stationary (WSS) if its mean and auto-correlation are not a function of time. That is
  - $m_x(t) = E[x(t)] = m$
  - $R_{xx}(t_1,t_2) = R_x(\tau)$, where $\tau = t_1 - t_2$

- If x(t) is WSS then:
  - $R_x(\tau) = R_x(-\tau)$
  - $|R_x(\tau)| <= |R_x(0)|$ (max is achieved at $\tau = 0$)

- The power content of a WSS process is:

$$P_x = E[\lim_{t\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt = \lim_{t\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_x(0)dt = R_x(0)$$

Power Spectrum of a random process

- **If x(t) is WSS then the power spectral density function is given by:**

$$S_x(f) = F[R_x(\tau)]$$

- **The total power in the process is also given by:**

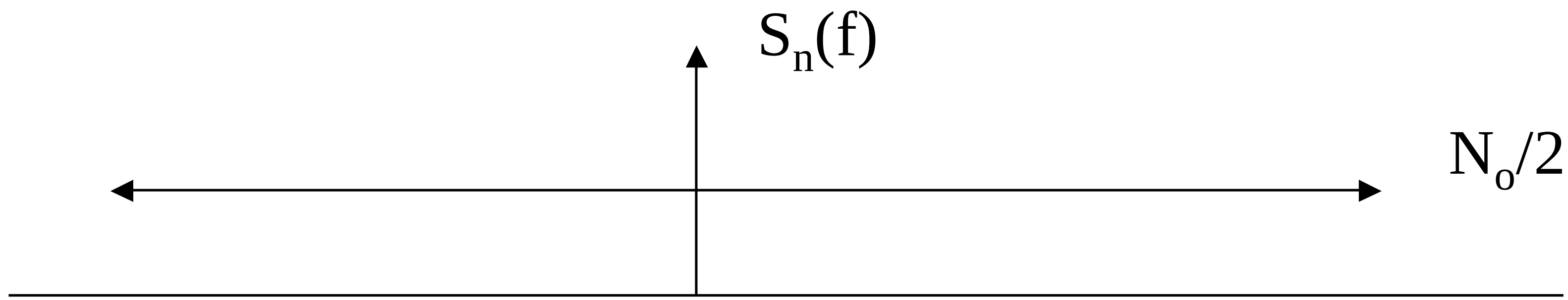
$$P_x = \int_{-\infty}^{\infty} S_x(f)df = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_x(t)e^{-j2\pi ft}dt\right]df$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R_x(t)e^{-j2\pi ft}df\right]dt$$

$$= \int_{-\infty}^{\infty} R_x(t)\left[\int_{-\infty}^{\infty} e^{-j2\pi ft}df\right]dt = \int_{-\infty}^{\infty} R_x(t)\delta(t)dt = R_x(0)$$

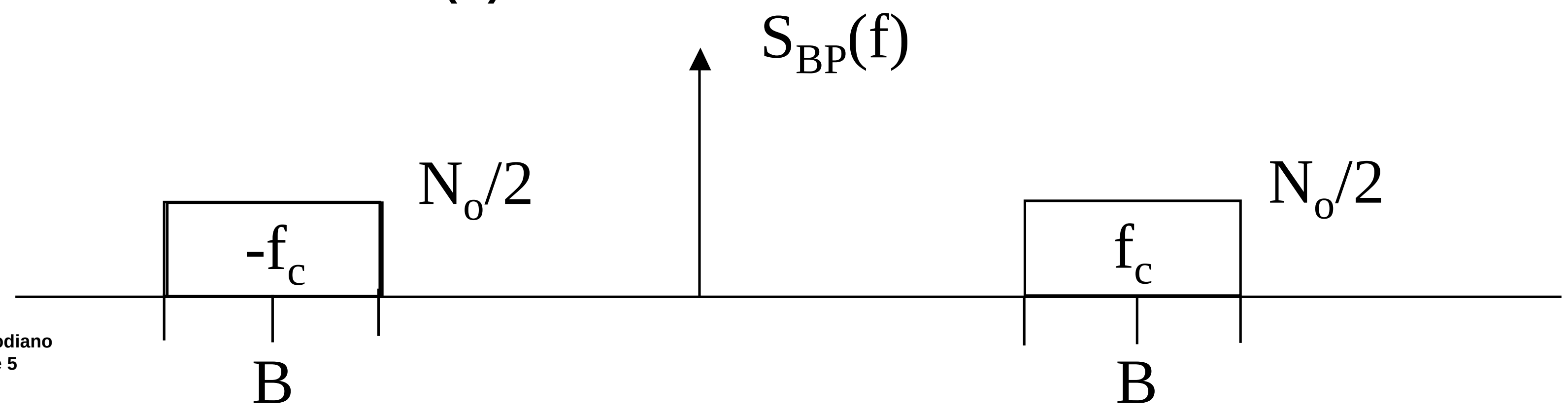
White noise

- The noise spectrum is flat over all relevant frequencies
  – White light contains all frequencies



- Notice that the total power over the entire frequency range is infinite
  – But in practice we only care about the noise content within the signal bandwidth, as the rest can be filtered out

- After filtering the only remaining noise power is that contained within the filter bandwidth (B)



AWGN

- **The effective noise content of bandpass noise is BN_o**
  - **Experimental measurements show that the pdf of the noise samples can be modeled as zero mean gaussian random variable**

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$$

  - **AKA Normal r.v., $N(0,\sigma^2)$**
  - **$\sigma^2 = P_x = BN_o$**

- **The CDF of a Gaussian R.V.,**

$$F_x(\alpha) = P[X \le \alpha] = \int_{-\infty}^{\alpha} f_x(x)dx = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2} dx$$

- **This integral requires numerical evaluation**
  - **Available in tables**

AWGN, continued

- $X(t) \sim N(0,\sigma^2)$
- $X(t_1)$, $X(t_2)$ are independent unless $t_1 = t_2$
-

$$R_x(\tau) = E[X(t+\tau)X(t)] = \begin{cases} E[X(t+\tau)]E[X(t)] & \tau \neq 0 \\ E[X^2(t)] & \tau = 0 \end{cases}$$

$$= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 & \tau = 0 \end{cases}$$

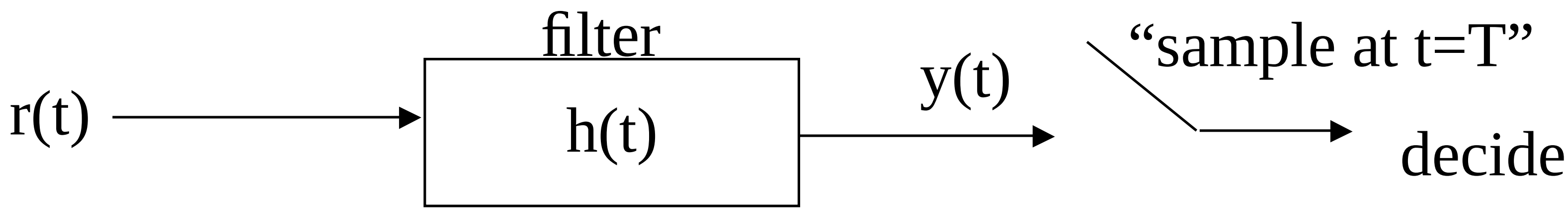
- $R_x(0) = \sigma^2 = P_x = BN_o$

Detection of signals in AWGN

Observe: $r(t) = S(t) + n(t), \ t \in [0,T]$

Decide which of $S_1, \ldots, S_m$ was sent

- **Receiver filter**
  - **Designed to maximize signal-to-noise power ratio (SNR)**



- **Goal: find h(t) that maximized SNR**

Receiver filter

$$y(t) = r(t) * h(t) = \int_0^t r(\tau)h(t-\tau)d\tau$$

Sampling at t = T $\Rightarrow$ $y(T) = \int_0^T r(\tau)h(T-\tau)d\tau$

$r(\tau) = s(\tau) + n(\tau) \Rightarrow$

$$y(T) = \int_0^T s(\tau)h(T-\tau)d\tau + \int_0^T n(\tau)h(T-\tau)d\tau = Y_s(T) + Y_n(T)$$

$$SNR = \frac{Y_s^2(T)}{E[Y_n^2(T)]} = \frac{\left[\int_0^T s(\tau)h(T-\tau)d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} = \frac{\left[\int_0^T h(\tau)s(T-\tau)d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt}$$

Matched filter: maximizes SNR

Caushy - Schwartz Inequality :

$$\left[\int_{-\infty}^{\infty} g_1(t) g_2(t) dt\right]^2 \le \int_{-\infty}^{\infty} (g_1(t))^2 \int_{-\infty}^{\infty} (g_2(t))^2$$

Above holds with equality iff: $g_1(t) = c g_2(t)$ for arbitrary constant c

$$SNR = \frac{\left[\int_0^T s(\tau) h(T-\tau) d\tau\right]^2}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} \le \frac{\int_0^T (s(\tau))^2 d\tau \int_0^T h^2(T-\tau) d\tau}{\frac{N_0}{2}\int_0^T h^2(T-t)dt} = \frac{2}{N_0}\int_0^T (s(\tau))^2 d\tau = \frac{2E_s}{N_0}$$

Above maximum is obtained iff: $h(T-\tau) = cS(\tau)$

$\Rightarrow h(t) = cS(T-t) = S(T-t)$

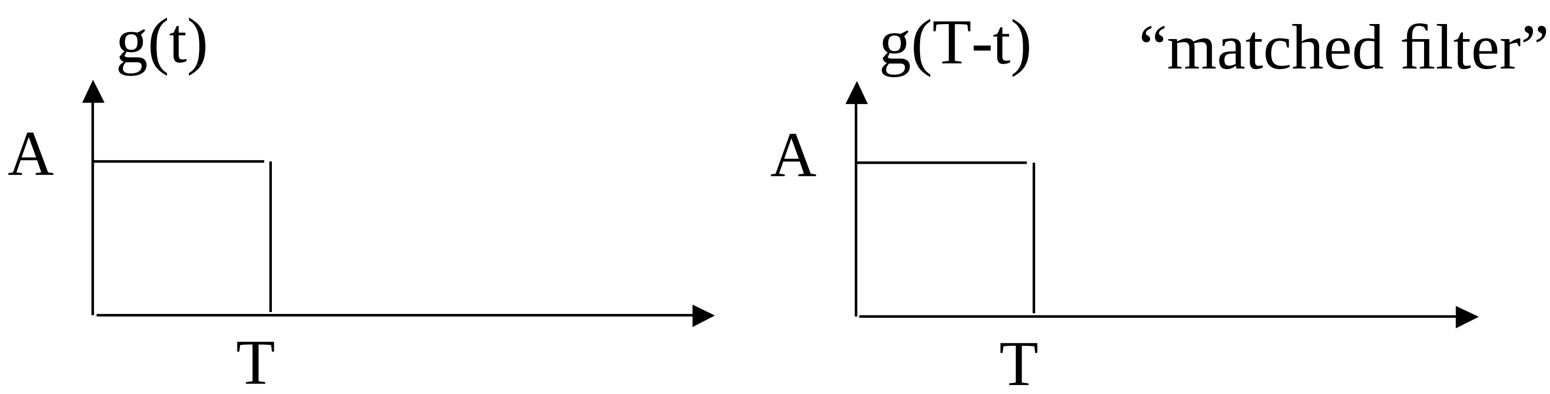
h(t) is said to be "matched" to the signal S(t)

Example: PAM

$S_m(t) = A_m g(t),\ \ t \in [0,T]$

A_m is a constant: Binary PAM $A_m \in \{0,1\}$

Matched filter is matched to g(t)

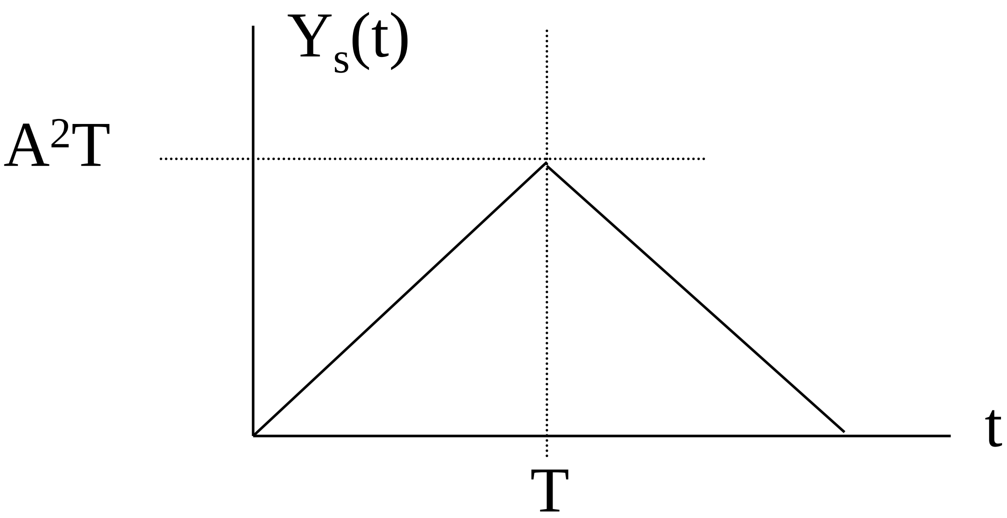


Example, continued

$$Y_s(t) = \int_0^t S(\tau)h(t-\tau)d\tau, \quad h(t') = g(T-t') \Rightarrow h(t-\tau) = g(T+\tau-t)$$

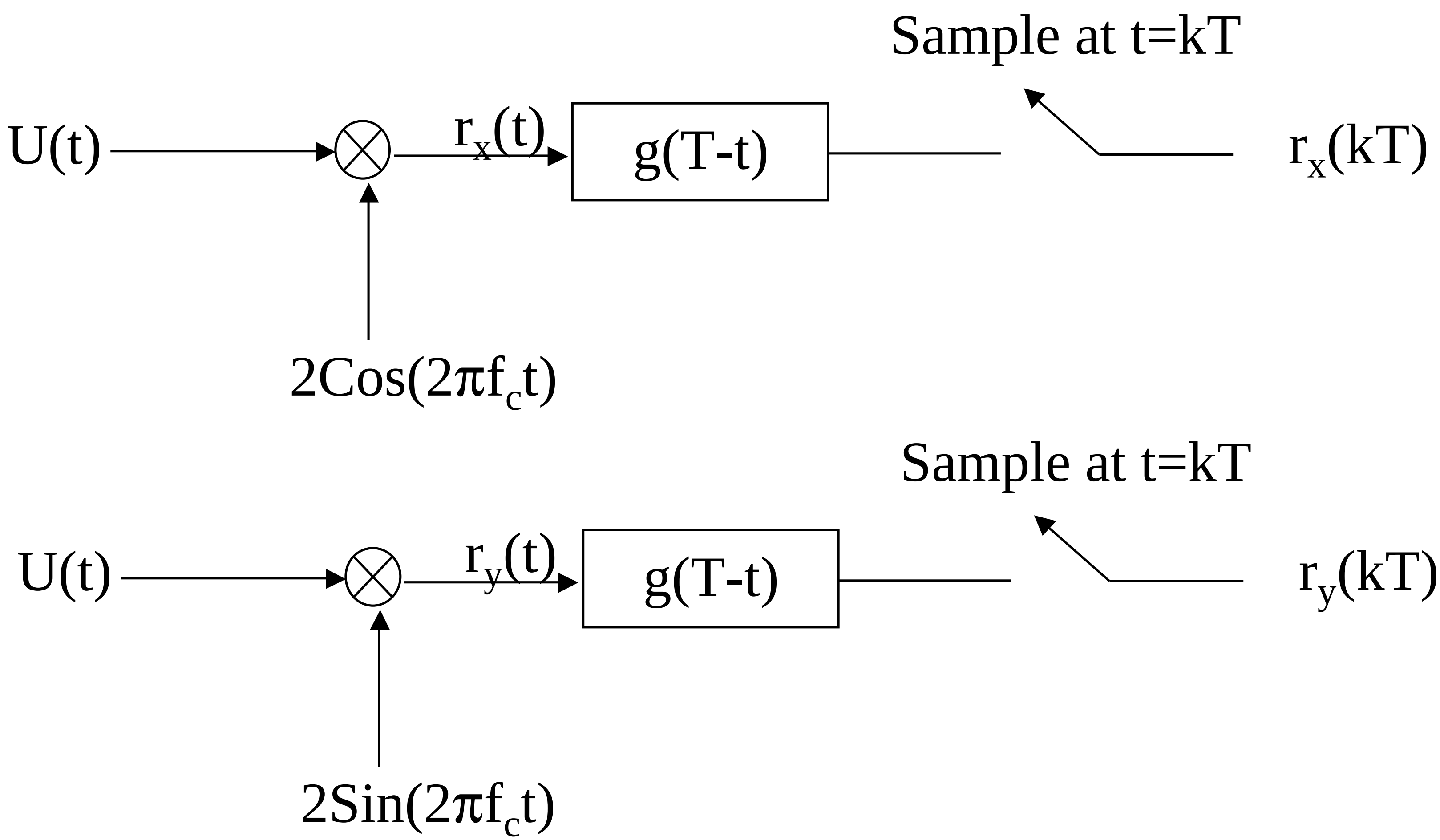
$$Y_s(t) = \int_0^t g(\tau)g(T+\tau-t)d\tau = \int_0^t g(\tau)g(T-t+\tau)d\tau$$

$$Y_s(T) = \int_0^T g^2(\tau)d\tau$$



- **Sample at t=T to obtain maximum value**

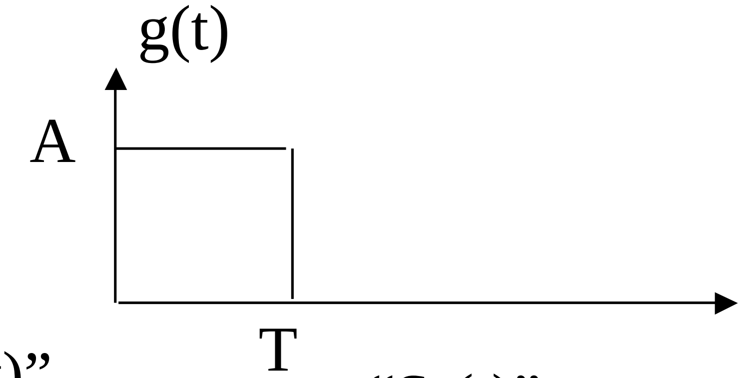
Matched filter receiver

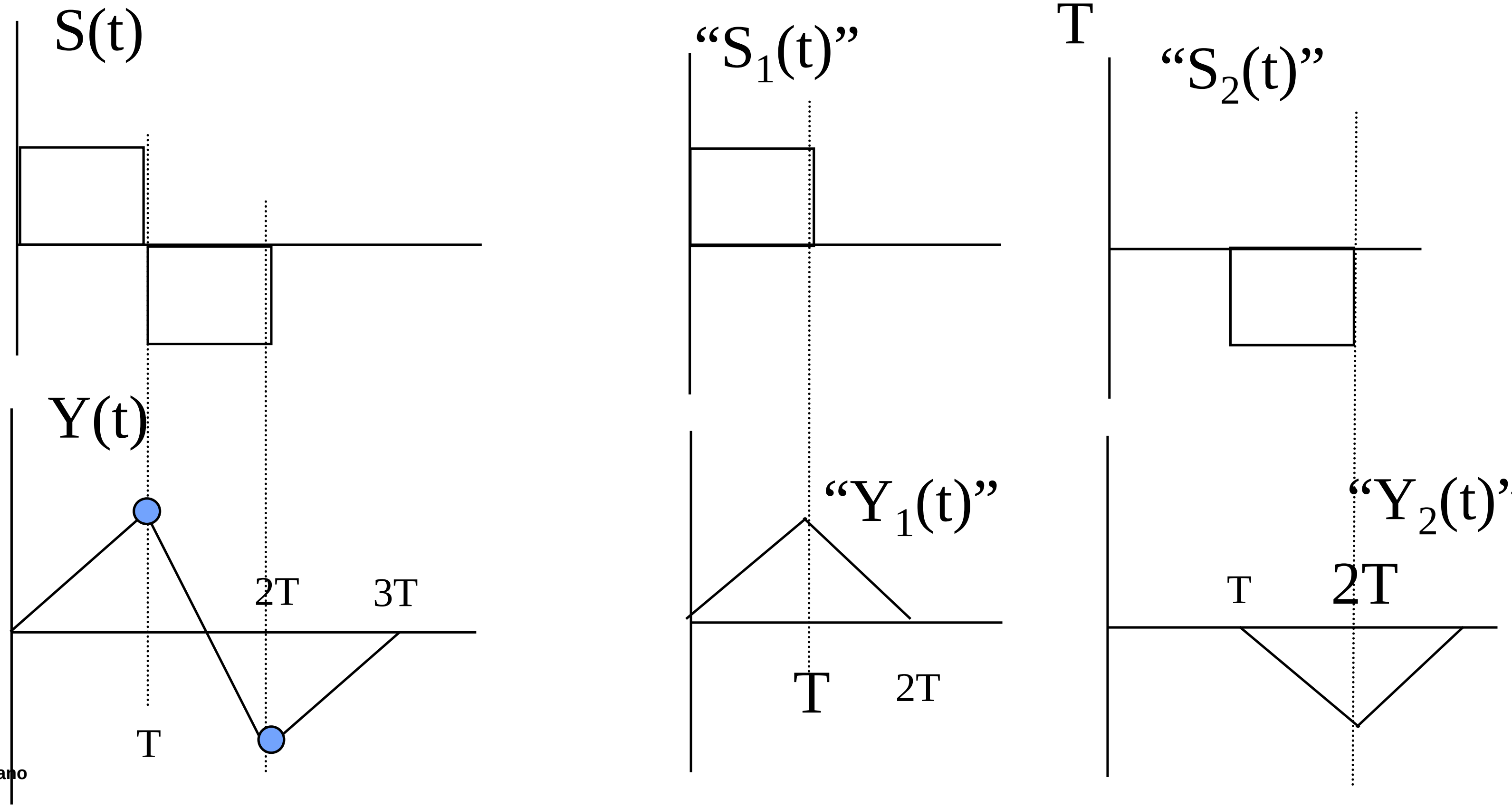


Binary PAM example, continued

$0 \Rightarrow S_1 = g(t)$

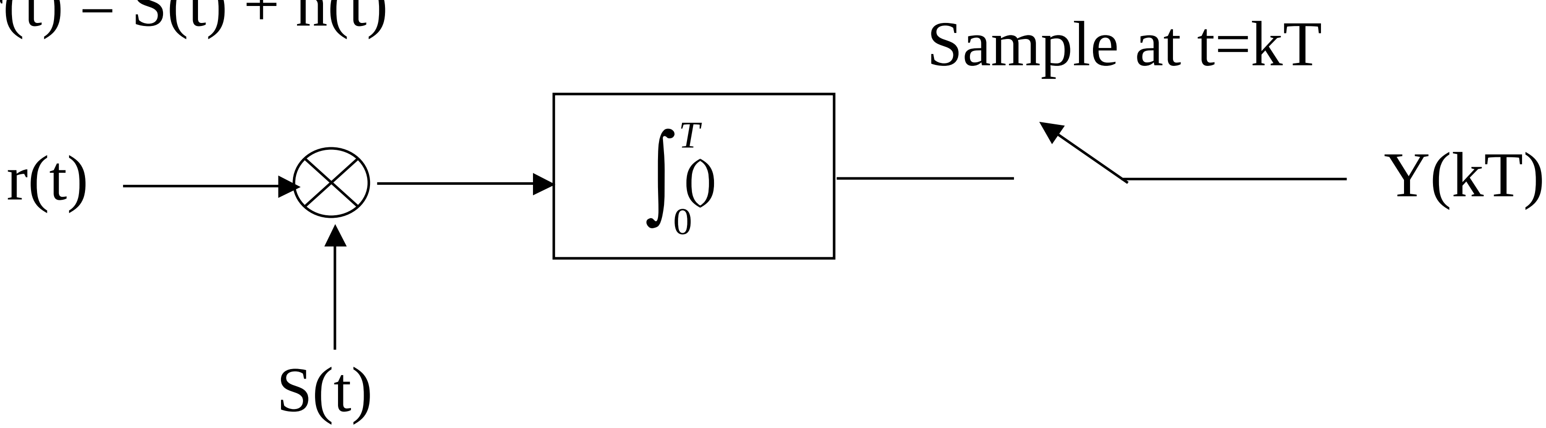
$1 \Rightarrow S_2 = -g(t)$





Alternative implementation: correlator receiver

r(t) = S(t) + n(t)



$$Y(T) = \int_0^T r(t)S(t) = \int_0^T S^2(t) + \int_0^T n(t)S(t) = Y_s(T) + Y_n(T)$$

Notice resemblance to matched filter