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16.36 Communication Systems Engineering
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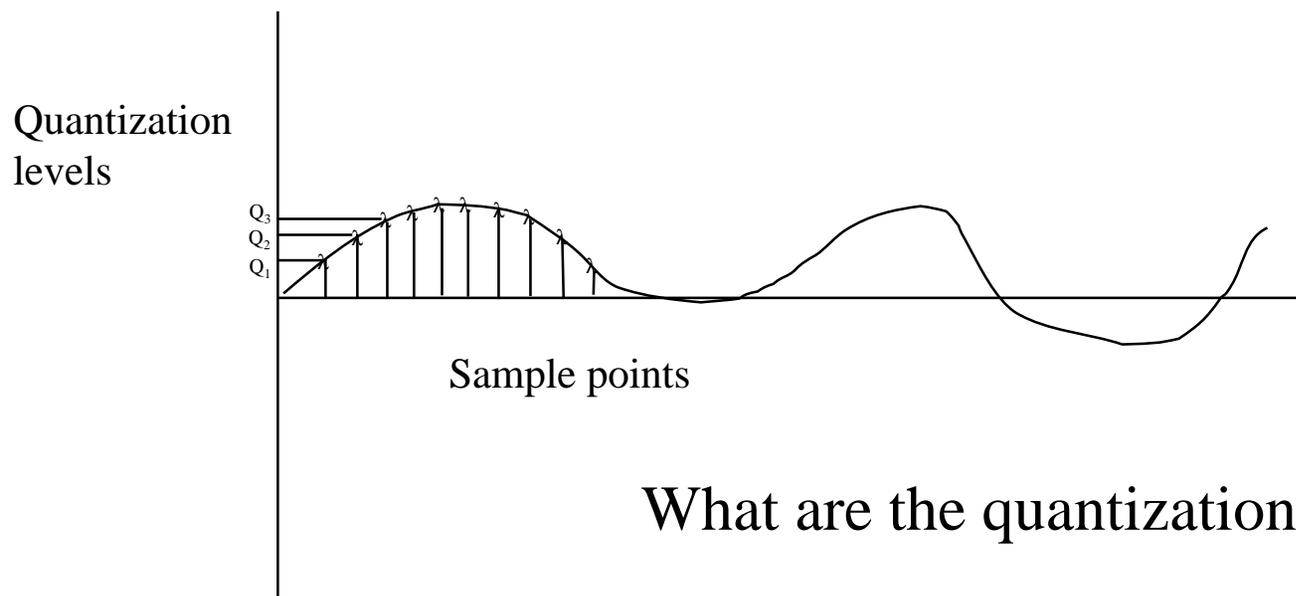
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Lecture 4: Quantization

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Sampling

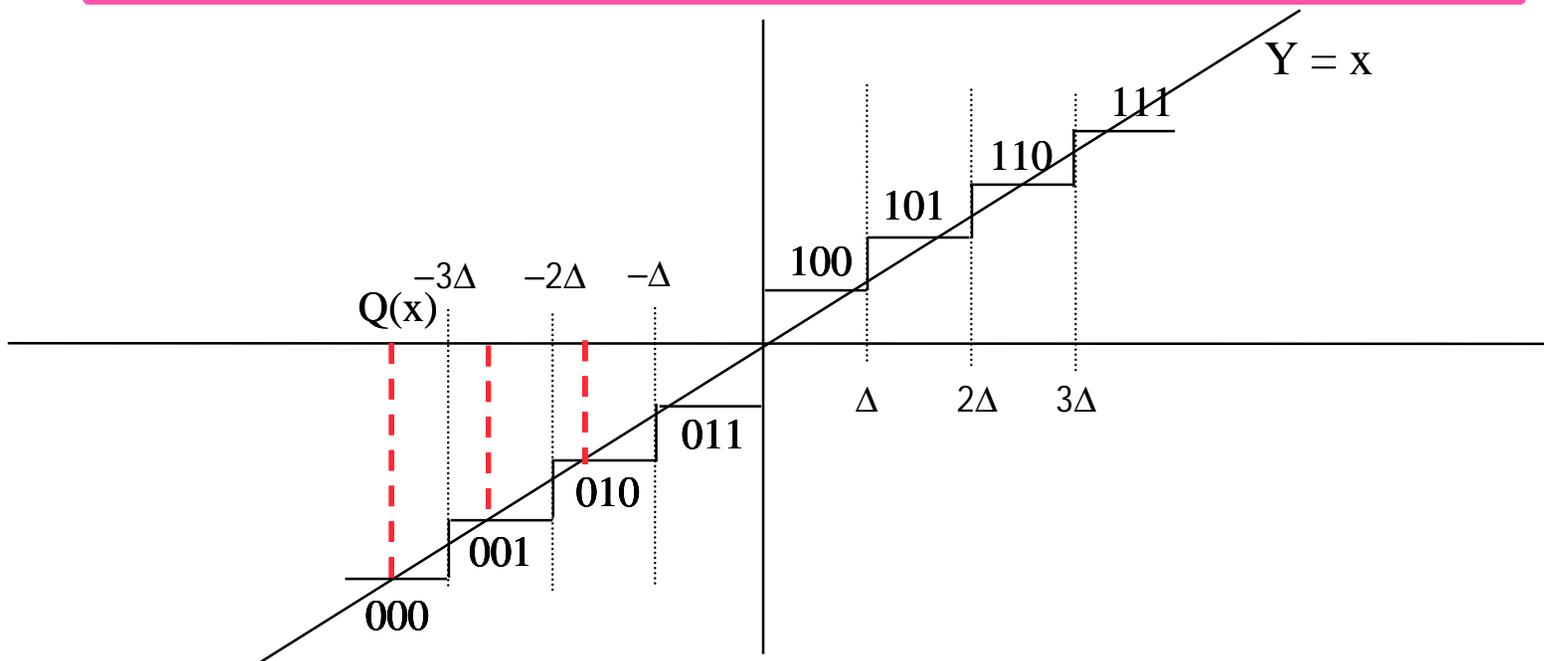
- **Sampling provides a discrete-time representation of a continuous waveform**
 - **Sample points are real-valued numbers**
 - **In order to transmit over a digital system we must first convert into discrete valued numbers**



What are the quantization regions

What are the quantization levels

Uniform Quantizer



- All quantization regions are of equal size (Δ)
 - Except first and last regions if samples are not finite valued
- With N quantization regions, use $\log_2(N)$ bits to represent each quantized value

Quantization Error

$$e(x) = Q(x) - x$$

$$\text{Squared error: } D = E[e(x)^2] = E[(Q(x)-x)^2]$$

$$\text{SQNR: } E[X^2]/E[(Q(x)-x)^2]$$

Example

- **X is uniformly distributed between -A and A**
 - $f(x) = 1/2A$, $-A \leq x \leq A$ and 0 otherwise
- **Uniform quantizer with N levels $\Rightarrow \Delta = 2A/N$**
 - $Q(x)$ = quantization level = midpoint of quantization region in which x lies
- **$D = E[e(x)^2]$ is the same for quantization regions**

$$D = E[e(x)^2 | x \in R_i] = \int_{-\Delta/2}^{\Delta/2} x^2 f(x) dx = \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} x^2 dx = \frac{\Delta^2}{12}$$

$$E[X] = \frac{1}{2A} \int_{-A}^A x^2 dx = \frac{A^2}{3}$$

$$SQNR = \frac{A^2/3}{\Delta^2/12} = \frac{A^2/3}{(2A/N)^2/12} = N^2, (\Delta = 2A/N)$$

Quantizer design

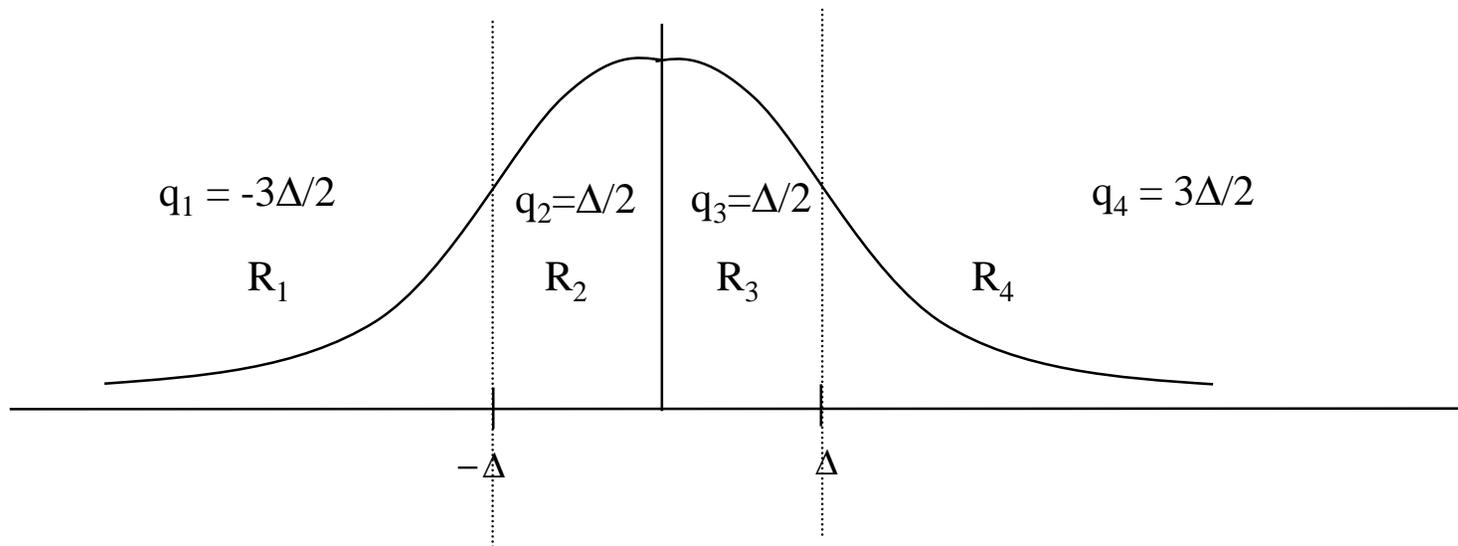
- **Uniform quantizer is good when input is uniformly distributed**
- **When input is not uniformly distributed**
 - **Non-uniform quantization regions**
Finer regions around more likely values
 - **Optimal quantization values not necessarily the region midpoints**
- **Approaches**
 - **Use uniform quantizer anyway**
Optimal choice of Δ
 - **Use non-uniform quantizer**
Choice of quantization regions and values
 - **Transform signal into one that looks uniform and use uniform quantizer**

Optimal uniform quantizer

- **Given the number of regions, N**
 - Find the optimal value of Δ
 - Find the optimal quantization values within each region
 - Optimization over $N+2$ variables
- **Simplification: Let quantization levels be the midpoint of the quantization regions (except first and last regions, when input not finite valued)**
- **Solve for Δ to minimize distortion**
 - Solution depends on input pdf and can be done numerically for commonly used pdfs (e.g., Gaussian pdf, table 6.2, p. 296 of text)

Uniform quantizer example

- **N=4, $X \sim N(0,1)$** $f_x(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma^2}, \sigma^2 = 1$
- **From table 6.2, $\Delta=0.9957$, $D=0.1188$, $H(Q)= 1.904$**
 - Notice that $H(Q)$ = the entropy of the quantized source is < 2
 - Two bits can be used to represent 4 quantization levels
 - Soon we will learn that you only need $H(Q)$ bits



Non-uniform quantizer

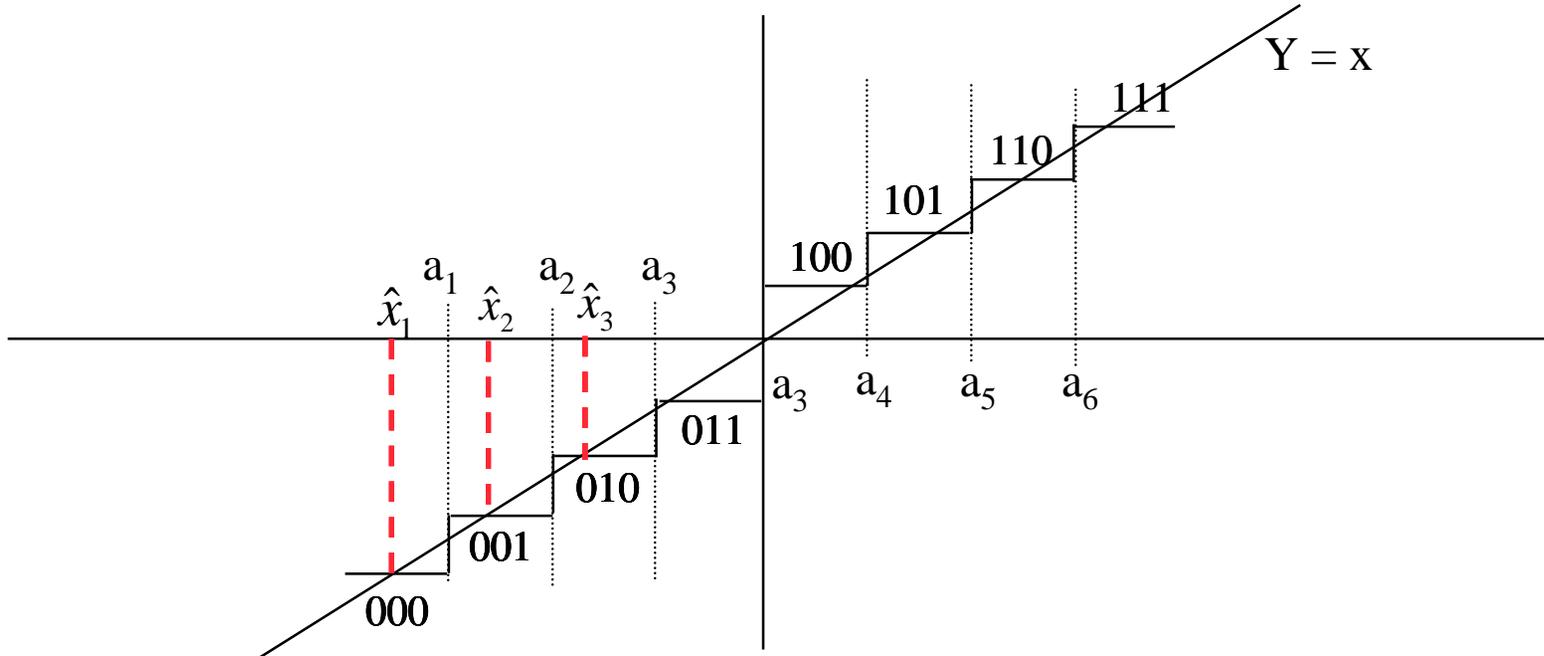
- Quantization regions need not be of same length
- Quantization levels need not be at midpoints
- Complex optimization over $2N$ variables

- Approach:
 - Given quantization regions, what should the quantization levels be?

 - What should the quantization regions be?

- Solve for quantization levels first (given region (a_{i-1}, a_i))
 - Minimize distortion

Non-uniform quantizer



Need to determine optimum quantization regions and levels

Optimal quantization levels

- **Minimize distortion, D**
 - **Optimal value affects distortion only within its region**

$$D_R = \int_{a_{i-1}}^{a_i} (x - \hat{x}_i)^2 f_x(x) dx$$

$$\frac{dD_R}{d\hat{x}_i} = \int_{a_{i-1}}^{a_i} 2(x - \hat{x}_i) f_x(x) dx = 0$$

$$\hat{x}_i = \int_{a_{i-1}}^{a_i} x f_x(x | a_{i-1} \leq x \leq a_i) dx$$

$$\hat{x}_i = E[X | a_{i-1} \leq x \leq a_i]$$

-
- **Quantization values should be the “centroid” of their regions**
 - **The conditional expected value of that region**
- **Approach can be used to find optimal quantization values for the uniform quantizer as well**

Optimal quantization regions

- **Take derivative of D with respect to a_i**
 - Take derivative with respect to integral boundaries

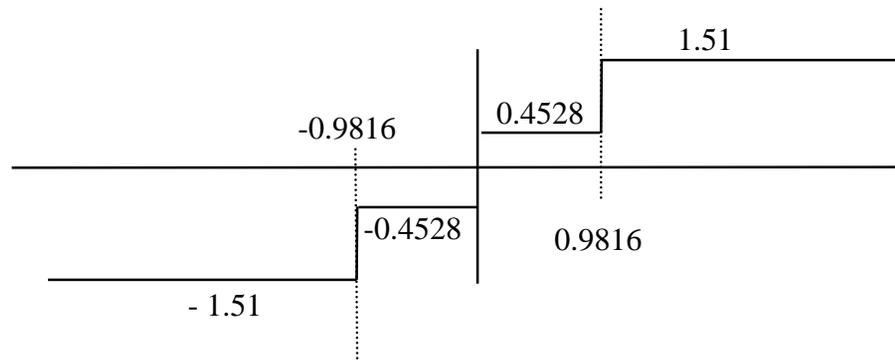
$$\frac{dD}{da_i} = f_x(a_i)[(a_i - \hat{x}_i)^2 - (a_i - \hat{x}_{i+1})^2] = 0$$

$$a_i = \frac{\hat{x}_i + \hat{x}_{i+1}}{2}$$

- Boundaries of the quantization regions are the midpoint of the quantization values
- **Optimality conditions:**
 1. Quantization values are the “centroid” of their region
 2. Boundaries of the quantization regions are the midpoint of the quantization values
 3. Clearly 1 depends on 2 and visa-versa. The two can be solved iteratively to obtain optimal quantizer

Finding the optimal quantizer

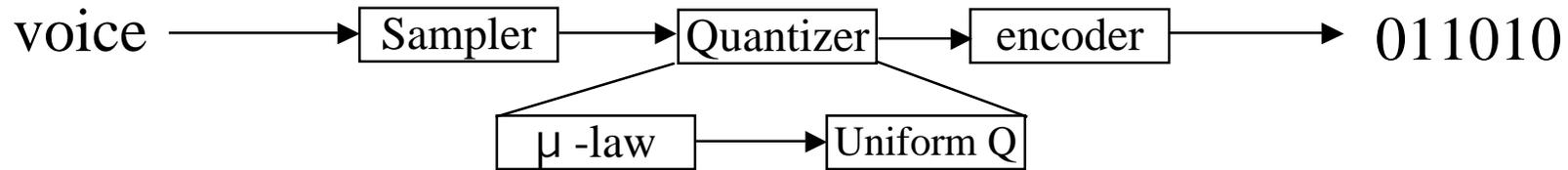
- Start with arbitrary regions (e.g., uniform Δ)
 - A) Find optimal quantization values (“centroids”)
 - B) Use quantization values to get new regions (“midpoints”)
 - Repeat A & B until convergence is achieved
- Can be done numerically for known distributions
 - Table 6.3 (p. 299) gives optimal quantizer for Gaussian source
- E.g., $N=4$,
 - $D = 0.1175$, $H(x) = 1.911$
 - Recall: uniform quantizer, $D= 0.1188$, $H(x) = 1.904$ (slight improvement)



Companders

- **Non-uniform quantizer can be difficult to design**
 - Requires knowledge of source statistics
 - Different quantizers for different input types
- **Solution: Transfer input signal into one that looks uniform and then use uniform quantizer**
- **Speech signal: high probabilities for low amplitudes**
 - Compress the large amplitudes before performing uniform quantization
- **μ -law compander**
$$g(x) = \frac{\text{Log}(1 + \mu |x|)}{\text{Log}(1 + \mu)} \text{sgn}(x)$$
 - μ controls the level of compression
 - $\mu = 255$ typically used for voice

Pulse code modulation



- **Uniform PCM: $x(t) \in [X_{min}, X_{max}]$**
 - $N = 2^v$ quantization levels, each level encoded using v bits
 - **SQNR: same as uniform quantizer**

$$SQNR = \frac{E[X^2] \times 3 \times 4^v}{X_{MAX}^2}$$

- **Notice that increasing the number of bits by 1 decreases SQNR by a factor of 4 (6 dB)**

Speech coding

- **PCM with $\mu = 255$**
- **Uniform quantizer with 128 levels, $N = 2^7$, 7 bits per sample**
- **Speech typically limited to 4KHZ**
 - **Sample at 8KHZ $\Rightarrow T_s = 1/8000 = 125 \mu s$**

8000 samples per second at 7 bits per sample $\Rightarrow 56$ Kbps
- **Differential PCM**
 - **Speech samples are typically correlated**
 - **Instead of coding samples independently, code the difference between samples**
 - **Result: improved performance, lower bit rate speech**