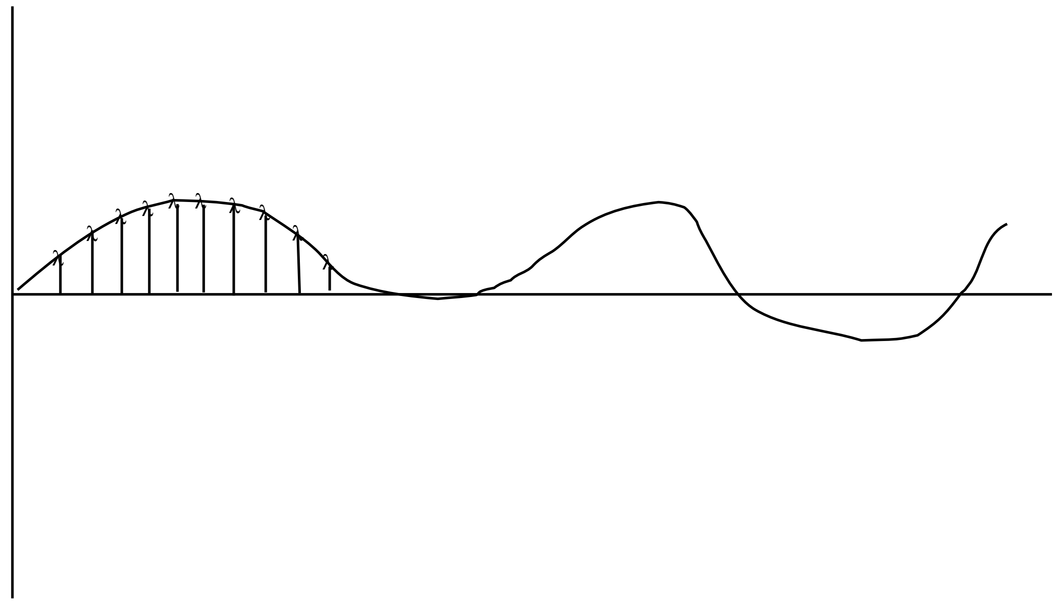
16.36 Communication Systems Engineering
Spring 2009

Lecture 2: The Sampling Theorem

Eytan Modiano

Sampling

- **Given a continuous time waveform, can we represent it using discrete samples?**
  - **How often should we sample?**
  - **Can we reproduce the original waveform?**



The Fourier Transform

- **Frequency representation of signals**
- **Definition:**

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

- **Notation:**

X(f) = F[x(t)]
X(t) = F-1 [X(f)]
x(t) ↔ X(f)

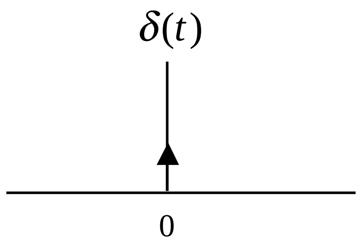
Unit impulse δ(t)

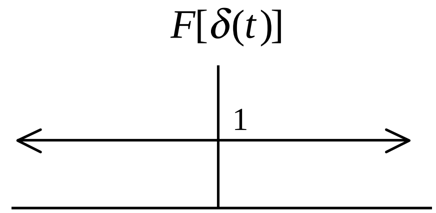
$$\delta(t) = 0, \forall t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t) = 1$$

$$\int_{-\infty}^{\infty} \delta(t)x(t) = x(0)$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)x(\tau) = x(t)$$



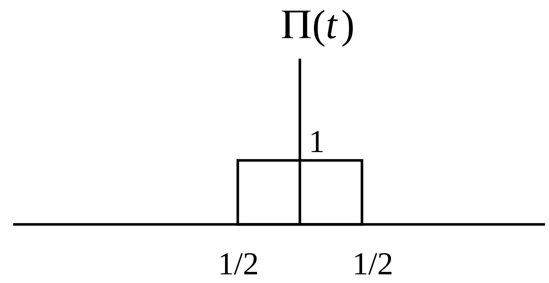


$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = e^{0} = 1$$

$$\delta(t) \Leftrightarrow 1$$

Rectangle pulse

$$\Pi(t) = \begin{cases} 1 & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0 & otherwise \end{cases}$$



$$F[\Pi(t)] = \int_{-\infty}^{\infty} \Pi(t) e^{-j2\pi ft} dt = \int_{-1/2}^{1/2} e^{-j2\pi ft} dt$$

$$= \frac{e^{-j\pi f} - e^{j\pi f}}{-j2\pi f} = \frac{Sin(\pi f)}{\pi f} = Sinc(f)$$

Properties of the Fourier transform

- **Linearity**
  - **$x1(t) <=> X1(f)$, $x2(t) <=> X2(f) => \alpha x1(t) + \beta x2(t) <=> \alpha X1(f) + \beta X2(f)$**
- **Duality**
  - **$X(f) <=> x(t) => x(f) <=> X(-t)$ and $x(-f) <=> X(t)$**
- **Time-shifting: $x(t-\tau) <=> X(f)e^{-j2\pi f\tau}$**
- **Scaling: $F[(x(at)] = 1/|a|\ X(f/a)$**
- **Convolution: $x(t) <=> X(f)$, $y(t) <=> Y(f)$ then,**
  - **$F[x(t)*y(t)] = X(f)Y(f)$**
  - **Convolution in time corresponds to multiplication in frequency and vice versa**

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(t-\tau) y(\tau) d\tau$$

Fourier transform properties (Modulation)

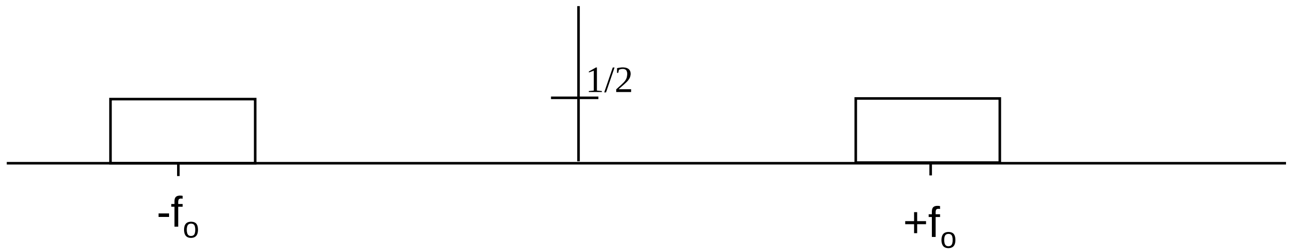
$$x(t)e^{j2\pi f_o t} \Leftrightarrow X(f - f_o)$$

$$Now,\ \cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$x(t)\cos(2\pi f_o t) = \frac{x(t)e^{j2\pi f_o t} + x(t)e^{-j2\pi f_o t}}{2}$$

$$Hence,\ x(t)\cos(2\pi f_o t) \Leftrightarrow \frac{X(f - f_o) + X(f + f_o)}{2}$$

- **Example: x(t)= sinc(t), F[sinc(t)] = Π(f)**
- **Y(t) = sinc(t)cos(2πf_ot) <=> (Π(f-f_o)+Π(f+f_o))/2**



More properties

- **Power content of signal**

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

- **Autocorrelation**

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t-\tau) dt$$

$$R_x(\tau) \Leftrightarrow |X(f)|^2$$

- **Sampling**

$$x(t_o) = x(t)\delta(t - t_o)$$

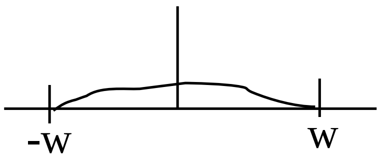
$$x(t) \sum_{n=-\infty}^{\infty} \delta(t - nt_o) = \text{sampled version of x(t)}$$

$$F[\sum_{n=-\infty}^{\infty} \delta(t - nt_o)] = \frac{1}{t_o} \sum_{n=-\infty}^{\infty} \delta(f - \frac{n}{t_o})]$$

The Sampling Theorem

- **Band-limited signal**
  - **Bandwidth < W**

$$X(f) = 0, \text{ for all } f, |f| \geq W$$



<u>Sampling Theorem:</u> If we sample the signal at intervals Ts where Ts ≤ 1/ 2W then signal can be completely reconstructed from its samples using the formula

$$x(t) = \sum_{n=-\infty}^{\infty} 2W' T_s x(nT_s) \text{sinc}[2W'(t - nT_s)]$$

$$Where, \quad W \leq W' \leq \frac{1}{T_s} - W$$

$$With T_s = \frac{1}{2W} \Rightarrow x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \text{sinc}[(\frac{t}{T_s} - n)]$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(\frac{n}{2W}) \text{sinc}[2W(t - \frac{n}{2W})]$$

Proof

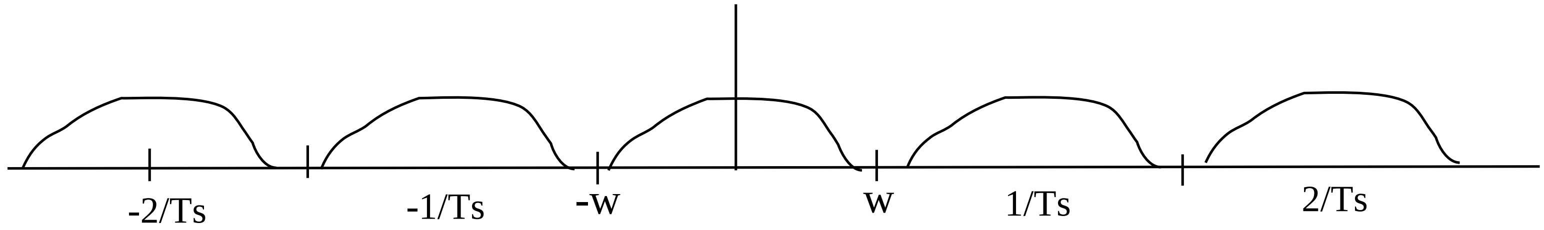
$$x_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t - nT_s)$$

$$X_\delta(f) = X(f) * F[\sum_{n=-\infty}^{\infty}\delta(t - nT_s)]$$

$$F[\sum_{n=-\infty}^{\infty}\delta(t - nT_s)] = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\delta(f - \frac{n}{T_s})$$

$$X_\delta(f) = \frac{1}{Ts}\sum_{n=-\infty}^{\infty}X(f - \frac{n}{T_s})$$

- **The Fourier transform of the sampled signal is a replication of the Fourier transform of the original separated by 1/Ts intervals**



Proof, continued

- If 1/Ts > 2W then the replicas of X(f) will not overlap and can be recovered

- How can we reconstruct the original signal?
  – Low pass filter the sampled signal

- Ideal low pass filter is rectangular
  – Its impulse response is a sinc function

$$H(f) = T_s \Pi(\frac{f}{2W})$$

- Now the recovered signal after low pass filtering

$$X(f) = X_\delta(f) T_s \Pi(\frac{f}{2W})$$

$$x(t) = F^{-1}[X_\delta(f) T_s \Pi(\frac{f}{2W})]$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) Sinc(\frac{t}{T_s} - n)$$

Notes about Sampling Theorem

- When sampling at rate 2W the reconstruction filter must be a rectangular pulse
  - Such a filter is not realizable
  - For perfect reconstruction must look at samples in the infinite future and past

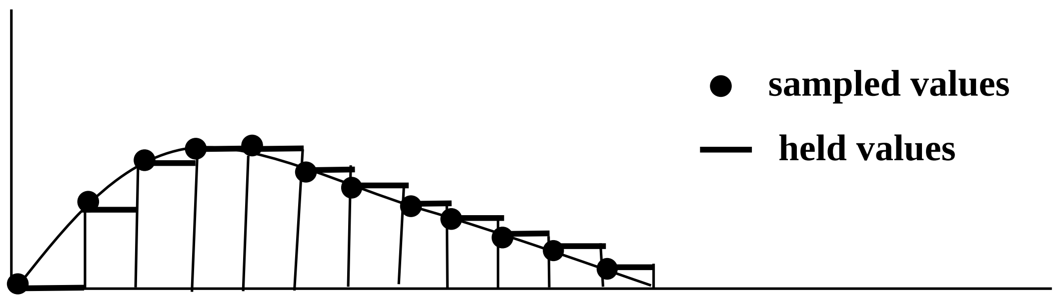
- In practice we can sample at a rate somewhat greater than 2W which makes reconstruction filters that are easier to realize

- Given any set of arbitrary sample points that are 1/2W apart, can construct a continuous time signal band-limited to W

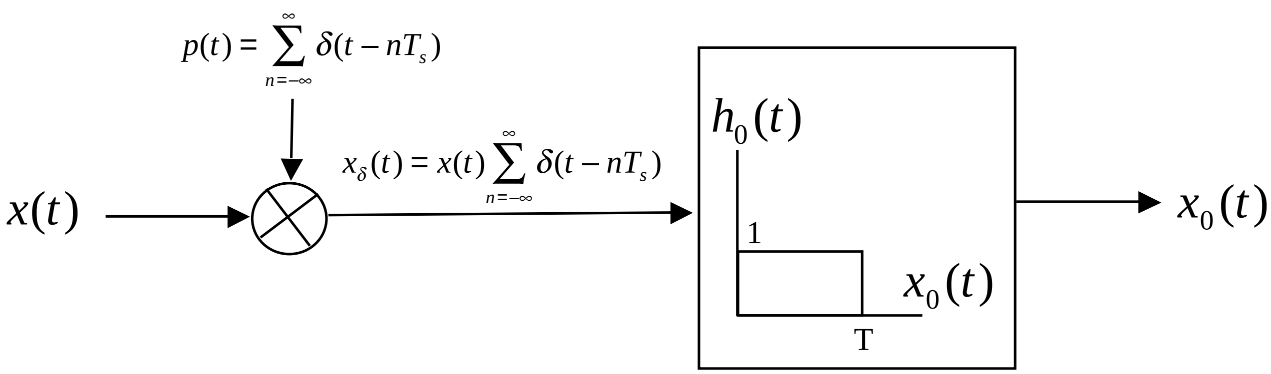
- Sampling using “impulses” is also not practical
  - Narrow pulses are difficult to implement
  - In practice, sampling is done using small rectangular pulses or “zero-order-hold”

Zero-Order Hold

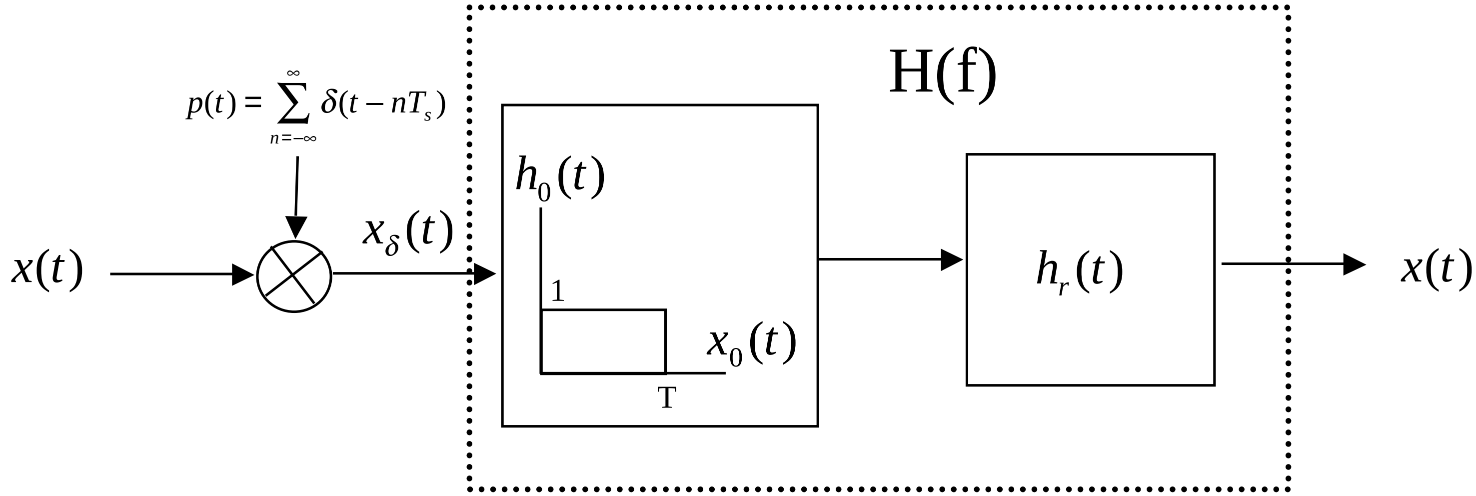
- **A form of "interpolation"**
- **The sampled signal holds its value until the next sample time**



- **In principle, zero-order hold can be realized with a cascade of an impulse train sampling and an LTI system with rectangular impulse response**



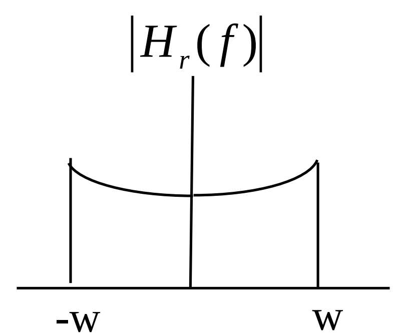
Reconstruction from zero-order hold



- **We know from the sampling theorem that in order to reconstruct x(t) from the impulse train samples on the left ($x_\delta(t)$) the filter on the right (H(f)) must be an ideal rectangular filter**

$$H(f) = T_s \Pi\left(\frac{f}{2W}\right) = T_s \Pi(T_s f)$$

$$\left.\begin{aligned} H(f) &= T_s \Pi\left(\frac{f}{2W}\right) = H_0(f) H_r(f) \\ H_0(f) &= e^{-j\pi f T}\left(\frac{\sin(\pi f T)}{\pi f}\right) \end{aligned}\right\} \Rightarrow H_r(f) = \frac{T_s \Pi\left(\frac{f}{2W}\right) e^{j\pi f T} \pi f}{\sin(\pi f T)}$$



Aliasing

- Sampling theorem requires that the signal be sampled at a frequency greater than twice its bandwidth

- When sampling at a frequency less than 2W, the replicas of the frequency spectrum overlap and cannot be “separated” using a low pass filter

- This is referred to as aliasing
  - Higher frequencies are “reflected” only lower frequencies
  - Signal cannot be recovered

- The term aliasing refers to the fact that the higher frequency signals become indistinguishable from the lower frequency ones