16.346 Astrodynamics
Fall 2008

Lagrange Multipliers Example

Consider a simple example of the use of Lagrange Multipliers:

Find the point on the curve $x^2y = 2$ which is nearest the origin.

Here we must make $x^2 + y^2$ a minimum subject to the constraint $x^2y - 2 = 0$.

Solution: Find the minimum of the function $f(x, y) = x^2 + y^2 - \lambda(x^2y - 2)$ when x and y are unconstrained:

$$\frac{\partial f}{\partial x} = 2x - \lambda 2xy = 0 \qquad \frac{\partial f}{\partial y} = 2y - \lambda x^2 = 0$$

from which we find: $\lambda = 1 \quad x = \pm\sqrt{2} \quad y = 1$ so that the two points at minimum distance from the origin are $\sqrt{2}, 1$ and $-\sqrt{2}, 1$.

Thrust Vector Attitude Control to Maximize Total Energy

State Equations

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ v_x \\ v_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ a_T \cos\beta(t) \\ a_T \sin\beta(t) - g \end{bmatrix} \iff \boxed{\frac{d\mathbf{x}}{dt} = \mathbf{f}[\mathbf{x}(t), \beta(t)]}$$

Performance Index

$$\boxed{J = gy(t_1) + \tfrac{1}{2}[v_x^2(t_1) + v_y^2(t_1)]} \iff J = gx_2(t_1) + \tfrac{1}{2}[x_3^2(t_1) + x_4^2(t_1)]$$

Admissible Variations

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}_m(t) + \alpha\boldsymbol{\epsilon}(t) \qquad \text{with} \qquad \boldsymbol{\epsilon}(t_0) = \mathbf{0} \\ \beta(t) &= \beta_m(t) + \alpha\gamma(t) \end{aligned}$$

Lagrange Multipliers

Introduce the vector Lagrange Multiplier $\boldsymbol{\lambda}(t)$ (also called the **Co-State**) and write

$$I = \int_{t_0}^{t_1} \boldsymbol{\lambda}^{\mathrm{T}}(t)\left(\frac{d\mathbf{x}}{dt} - \mathbf{f}[\mathbf{x}(t), \beta(t)]\right) dt = 0$$

The Problem

To maximize $J - I$ as a function of α

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = g\epsilon_2(t_1) + x_{3m}(t_1)\epsilon_3(t_1) + x_{4m}(t_1)\epsilon_4(t_1)$$

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = \int_{t_0}^{t_1} \boldsymbol{\lambda}^{\mathrm{T}}(t)\left(\frac{d\boldsymbol{\epsilon}}{dt} - \frac{\partial\mathbf{f}}{\partial\mathbf{x}}\boldsymbol{\epsilon} - \frac{\partial\mathbf{f}}{\partial\beta}\gamma\right) dt$$

Integration by Parts

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = \int_{t_0}^{t_1} \boldsymbol{\lambda}^{\mathrm{T}}(t)\Big(\frac{d\boldsymbol{\epsilon}}{dt} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\boldsymbol{\epsilon} - \frac{\partial \mathbf{f}}{\partial \beta}\gamma\Big)\,dt$$

$$= \boldsymbol{\lambda}^{\mathrm{T}}(t)\boldsymbol{\epsilon}(t)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\Big(\frac{d\boldsymbol{\lambda}^{\mathrm{T}}}{dt} + \lambda^{\mathrm{T}}(t)\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\Big)\boldsymbol{\epsilon}(t)\,dt - \int_{t_0}^{t_1} \boldsymbol{\lambda}^{\mathrm{T}}(t)\frac{\partial \mathbf{f}}{\partial \beta}\gamma(t)\,dt$$

$$= \boldsymbol{\lambda}^{\mathrm{T}}(t_1)\boldsymbol{\epsilon}(t_1) - \int_{t_0}^{t_1} \boldsymbol{\lambda}^{\mathrm{T}}(t)\frac{\partial \mathbf{f}}{\partial \beta}\gamma(t)\,dt \qquad \text{must equal} \qquad \left.\frac{dJ}{d\alpha}\right|_{\alpha=0}$$

Here we require the Co-State $\boldsymbol{\lambda}(t)$ to satisfy the differential equation

$$\boxed{\frac{d\boldsymbol{\lambda}^{\mathrm{T}}}{dt} = -\boldsymbol{\lambda}^{\mathrm{T}}\frac{\partial \mathbf{f}}{\partial \mathbf{x}}}$$

In our case

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{so that} \qquad \begin{aligned} \lambda_1(t) &= c_1 \\ \lambda_2(t) &= c_2 \\ \lambda_3(t) &= c_1 t + c_3 \\ \lambda_4(t) &= c_2 t + c_4 \end{aligned} \qquad \text{Also} \qquad \frac{\partial \mathbf{f}}{\partial \beta} = \begin{bmatrix} 0 \\ 0 \\ -a_T \sin\beta \\ a_T\cos\beta \end{bmatrix}$$

Choose the constants c_1, c_2, c_3, c_4 so that

$$\begin{aligned} \lambda_1(t) &= 0 & \qquad \lambda_3(t) &= v_{xm}(t_1) \\ \lambda_2(t) &= g & \qquad \lambda_4(t) &= g(t_1 - t) + v_{ym}(t_1) \end{aligned}$$

Then, if we are to have

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} - \left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0 \quad \text{we must require that} \quad \int_{t_0}^{t_1} \boldsymbol{\lambda}^{\mathrm{T}}(t)\frac{\partial \mathbf{f}}{\partial \beta}\gamma(t)\,dt = 0$$

From the **Fundamental Lemma of the Calculus of Variations** it follows that

$$\boxed{\boldsymbol{\lambda}^{\mathrm{T}}(t)\frac{\partial \mathbf{f}}{\partial \beta} = 0}$$

which is called the **Optimality Condition**.

In our case, we have

$$-\lambda_3(t)\sin\beta_m(t) + \lambda_4(t)\cos\beta_m(t) = 0$$

Thus, the optimum program for $\beta(t)$ is

$$\frac{\lambda_4(t)}{\lambda_3(t)} = \boxed{\tan\beta_m(t) = \frac{g(t_1 - t) + v_{ym}(t_1)}{v_{xm}(t_1)}}$$

called the **Linear-Tangent Law**.

This result formed the basis of the so-called **Iterated Guidance Mode** used by the Saturn launch vehicle's guidance system to place the Apollo spacecraft in an earth parking orbit prior to its voyage to the moon.