16.346 Astrodynamics
Fall 2008

Lecture 31 The Calculus of Variations & Lunar Landing Guidance

The Brachistochrone Problem

In a vertical xy-plane a smooth curve $y = f(x)$ connects the origin with a point $P(x_1, y_1)$ in such a way that the time taken by a particle sliding without friction from O to P along the curve propelled by gravity is as short as possible. What is the curve?

Assume the positive y-axis is vertically downward. Then the equation of motion is

$$m\frac{d^2 s}{dt^2} = mg\sin\gamma = mg\frac{dy}{ds} \quad \text{with} \quad ds^2 = dx^2 + dy^2$$

$$\frac{d^2 s}{dt^2}\frac{ds}{dt} = g\frac{dy}{ds}\frac{ds}{dt}$$

$$\frac{d}{dt}\left(\frac{ds}{dt}\right)^2 = 2g\frac{dy}{dt} \qquad \Longrightarrow \qquad \frac{ds}{dt} = \sqrt{2gy}$$

Then

$$\int_0^T dt = T = \int_0^{x_1} \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}}\int_0^{x_1} \frac{\sqrt{1+y'^2}}{\sqrt{y}}\,dx = \frac{1}{\sqrt{2g}}\int_0^{x_1} F(y, y')\,dx$$

Deriving Euler's Equation

To minimize the integral $\quad I = \int_{x_0}^{x_1} F(x, y, y')\,dx \quad$ let $\quad y(x,\alpha) = y_m(x) + \alpha\epsilon(x)$

Then
$$\frac{dI}{d\alpha} = \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y}\epsilon(x) + \frac{\partial F}{\partial y'}\frac{d\epsilon}{dx}\right]dx = \int_{x_0}^{x_1}\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right]\epsilon(x)\,dx$$

Therefore, from the **Fundamental Lemma of the Calculus of Variations**

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0}$$

is a Necessary Condition which F must satisfy if the integral I is to be a minimum.

Special Case of Euler's Equation

Also
$$\frac{d}{dx}\left(F - \frac{\partial F}{\partial y'}y'\right) = \frac{\partial F}{\partial x} + \underbrace{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)}_{=0} y' = \frac{\partial F}{\partial x}$$

which will be zero if F is **not** a function of x. Therefore

$$\boxed{F - \frac{\partial F}{\partial y'}y' = \text{constant}}$$

Prob. 11–33

which establishes the necessary condition used to solve the Brachistochrone Problem.

Solution of the Brachistochrone Problem

If T is to be a minimum, then, using Euler's Special Case of the Necessary Condition, we have

$$y(1+{y'}^2) = 2c \qquad \text{or} \qquad \int dx = x = \int \sqrt{\frac{y}{2c-y}}\,dy$$

Now let

$$y = 2c\sin^2\theta = c(1-\cos 2\theta)$$

so that

$$x = 2c\int (1-\cos 2\theta)\,d\theta = c(2\theta - \sin 2\theta)$$

Therefore, the equation of the curve in parametric form is

$$\boxed{\begin{aligned} x &= c(\phi - \sin\phi) \\ y &= c(1-\cos\phi) \end{aligned}} \qquad \text{with} \qquad \phi = 2\theta$$

and represents a cycloid—the path of a point on a circle of radius c as it rolls along the underside of the x axis.

Terminal State Vector Control

Find the acceleration vector $\mathbf{a}(t)$ to minimize

$$\boxed{J = \int_{t_0}^{t_1} a(t)^2\,dt = \int_{t_0}^{t_1} \mathbf{a}^{\mathrm{T}}(t)\mathbf{a}(t)\,dt}$$

subject to

$$\begin{aligned} \frac{d\mathbf{r}}{dt} &= \mathbf{v} \qquad & \mathbf{r}(t_0) &= \mathbf{r}_0 \qquad & \mathbf{r}(t_1) &= \mathbf{r}_1 \\ \frac{d\mathbf{v}}{dt} &= \mathbf{a} \qquad & \mathbf{v}(t_0) &= \mathbf{v}_0 \qquad & \mathbf{v}(t_1) &= \mathbf{v}_1 \end{aligned}$$

Define the Admissible Functions:

$$\begin{aligned} \mathbf{r}(t,\alpha) &= \mathbf{r}_m(t) + \alpha\boldsymbol{\delta}(t) & & \boldsymbol{\delta}(t_0) = \boldsymbol{\delta}(t_1) = \mathbf{0} \\ \mathbf{v}(t,\alpha) &= \mathbf{v}_m(t) + \alpha\boldsymbol{\delta}'(t) \qquad & \text{where} \qquad & \boldsymbol{\delta}'(t_0) = \boldsymbol{\delta}'(t_1) = \mathbf{0} \\ \mathbf{a}(t,\alpha) &= \mathbf{a}_m(t) + \alpha\boldsymbol{\delta}''(t) & & \boldsymbol{\delta}''(t_0) = \boldsymbol{\delta}''(t_1) = \mathbf{0} \end{aligned}$$

Then

$$J(\alpha) = \int_{t_0}^{t_1} \mathbf{a}_m^{\mathrm{T}}(t)\mathbf{a}_m(t)\,dt + 2\alpha\int_{t_0}^{t_1} \mathbf{a}_m^{\mathrm{T}}(t)\,\boldsymbol{\delta}''(t)\,dt + \alpha^2\int_{t_0}^{t_1} \boldsymbol{\delta}''(t)^{\mathrm{T}}\boldsymbol{\delta}''(t)\,dt$$

A Necessary Condition for

$$J(\alpha) = \int_{t_0}^{t_1} \mathbf{a}_m^{\mathsf{T}}(t)\mathbf{a}_m(t)\,dt + 2\alpha \int_{t_0}^{t_1} \mathbf{a}_m^{\mathsf{T}}(t)\,\boldsymbol{\delta}''(t)\,dt + \alpha^2 \int_{t_0}^{t_1} \boldsymbol{\delta}''(t)^{\mathsf{T}}\boldsymbol{\delta}''(t)\,dt$$

to be a minimum is that

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0 = 2\int_{t_0}^{t_1} \mathbf{a}_m^{\mathsf{T}}(t)\,\boldsymbol{\delta}''(t)\,dt$$

Use integration by parts

$$\int_{t_0}^{t_1} \mathbf{a}_m^{\mathsf{T}}(t)\boldsymbol{\delta}''\,dt = \mathbf{a}_m^{\mathsf{T}}(t)\boldsymbol{\delta}'(t)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d\mathbf{a}_m^{\mathsf{T}}(t)}{dt}\frac{d\boldsymbol{\delta}(t)}{dt}\,dt = 0 - \int_{t_0}^{t_1} \frac{d\mathbf{a}_m^{\mathsf{T}}(t)}{dt}\frac{d\boldsymbol{\delta}(t)}{dt}\,dt$$

$$= -\frac{d\mathbf{a}_m^{\mathsf{T}}(t)}{dt}\boldsymbol{\delta}(t)\Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \frac{d^2\mathbf{a}_m^{\mathsf{T}}(t)}{dt^2}\boldsymbol{\delta}(t)\,dt = 0 + \int_{t_0}^{t_1} \frac{d^2\mathbf{a}_m^{\mathsf{T}}(t)}{dt^2}\boldsymbol{\delta}(t)\,dt$$

Hence

$$\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0 \quad\Longrightarrow\quad \boxed{\int_{t_0}^{t_1} \frac{d^2\mathbf{a}_m^{\mathsf{T}}(t)}{dt^2}\,\boldsymbol{\delta}(t)\,dt = 0}$$

Again using the **Fundamental Lemma of the Calculus of Variations** it follows that

$$\frac{d^2\mathbf{a}_m^{\mathsf{T}}(t)}{dt^2} = \mathbf{0}^{\mathsf{T}} \qquad\Longrightarrow\qquad \mathbf{a}_m(t) = \mathbf{c}_1 t + \mathbf{c}_2$$

Therefore, with $t_{go} = t_1 - t$, we have

$$\boxed{\mathbf{a}_m(t) = \mathbf{c}_1 t + \mathbf{c}_2 = \frac{4}{t_{go}}[\mathbf{v}_1 - \mathbf{v}(t)] + \frac{6}{t_{go}^2}\{\mathbf{r}_1 - [\mathbf{r}(t) + \mathbf{v}_1 t_{go}]}$$

Lunar-Landing Guidance for Apollo Missions

To include the effects of gravity

$$\mathbf{a}(t) = \mathbf{a}_T(t) + \mathbf{g}(\mathbf{r})$$

we could use

$$\boxed{\mathbf{a}_T(t) = \frac{4}{t_{go}}[\mathbf{v}_1 - \mathbf{v}(t)] + \frac{6}{t_{go}^2}\{\mathbf{r}_1 - [\mathbf{r}(t) + \mathbf{v}_1 t_{go}]\} - \mathbf{g}[\mathbf{r}(t)]}$$

for the thrust acceleration which would be an exact solution if $\mathbf{g}$ were constant.