16.346 Astrodynamics
Fall 2008

The Disturbing Function

Three-body equations of motion

$$\frac{d^2\mathbf{r}_1}{dt^2} = G\frac{m_2}{r_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_3}{r_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1)$$
$$\frac{d^2\mathbf{r}_2}{dt^2} = G\frac{m_1}{r_{21}^3}(\mathbf{r}_1 - \mathbf{r}_2) + G\frac{m_3}{r_{23}^3}(\mathbf{r}_3 - \mathbf{r}_2)$$
$$\frac{d^2\mathbf{r}_3}{dt^2} = G\frac{m_1}{r_{31}^3}(\mathbf{r}_1 - \mathbf{r}_3) + G\frac{m_2}{r_{32}^3}(\mathbf{r}_2 - \mathbf{r}_3)$$

Ignore the third equation. Subtract the first equation from the second and define

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \qquad \boldsymbol{\rho} = \mathbf{r}_3 - \mathbf{r}_1 \qquad \mathbf{d} = \mathbf{r} - \boldsymbol{\rho} \qquad \mu = G(m_1 + m_2) \qquad m = m_3$$

Then, the equation of relative motion may be written as

$$\frac{d^2\mathbf{r}^{\mathsf{T}}}{dt^2} + \frac{\mu}{r^3}\mathbf{r}^{\mathsf{T}} = -Gm\left(\frac{1}{d^3}\mathbf{d}^{\mathsf{T}} + \frac{1}{\rho^3}\boldsymbol{\rho}^{\mathsf{T}}\right) = Gm\frac{\partial}{\partial\mathbf{r}}\left(\frac{1}{d} - \frac{1}{\rho^3}\mathbf{r}\cdot\boldsymbol{\rho}\right) = \frac{\partial R}{\partial\mathbf{r}}$$

where R, called the disturbing function, can be written as

$$R = Gm\left(\frac{1}{d} - \frac{1}{\rho^3}\mathbf{r}\cdot\boldsymbol{\rho}\right) = Gm\left(\frac{1}{d} - \frac{1}{\rho^3}r\rho\cos\alpha\right) = \frac{Gm}{\rho}\left(\frac{\rho}{d} - \overbrace{\frac{r}{\rho}}^{x}\underbrace{\cos\alpha}_{\nu}\right) = \frac{Gm}{\rho}\left(\frac{\rho}{d} - \nu x\right)$$

Since

$$\frac{d^2}{\rho^2} = \frac{(\mathbf{r}-\boldsymbol{\rho})\cdot(\mathbf{r}-\boldsymbol{\rho})}{\boldsymbol{\rho}\cdot\boldsymbol{\rho}} = \frac{r^2 - 2r\rho\cos\alpha + \rho^2}{\rho^2} = 1 - 2\nu x + x^2 \quad \Longrightarrow \quad \frac{\rho}{d} = (1 - 2\nu x + x^2)^{-\frac{1}{2}}$$

Generating Function for Legendre Polynomials

$$\mathcal{L}(x,\nu) = (1 - 2\nu x + x^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} P_k(\nu)x^k$$

The Disturbing Function and Its Gradient

$$\boxed{R = \frac{Gm}{\rho}\left[1 + \sum_{k=2}^{\infty} P_k(\cos\alpha)\left(\frac{r}{\rho}\right)^k\right]}$$

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r} = G\frac{m}{\rho^2}\sum_{k=1}^{\infty}\left(\frac{r}{\rho}\right)^k [P'_{k+1}(\cos\alpha)\,\mathbf{i}_\rho - P'_k(\cos\alpha)\,\mathbf{i}_r]$$

Properties of Legendre Polynomials

$$P_0(\nu) = 1$$
$$P_1(\nu) = \nu$$
$$P_2(\nu) = \tfrac{1}{2}(3\nu^2 - 1)$$
$$P_3(\nu) = \tfrac{1}{2}(5\nu^3 - 3\nu)$$
$$P_4(\nu) = \tfrac{1}{8}(35\nu^4 - 30\nu^2 + 3)$$

$$P_n(\nu) = \underbrace{F[-n, n+1; 1; \tfrac{1}{2}(1-\nu)]}_{\text{Hypergeometric function}} = \overbrace{\frac{1}{2^n n!}\frac{d^n}{d\nu^n}(\nu^2 - 1)^n}^{\text{Rodrigues' formula}}$$

$$\underbrace{nP_n(\nu) - (2n-1)\nu P_{n-1}(\nu) + (n-1)P_{n-2}(\nu) = 0}_{\text{Recursion formula}}$$

$$\left.\int_{-1}^{1} P_m(\nu)P_n(\nu)\,d\nu = 0 \right\} \quad \text{Orthogonality property}$$

$$(n-1)P_n'(\nu) - (2n-1)\nu P_{n-1}'(\nu) + nP_{n-2}'(\nu) = 0$$

Laplace's Sphere of Influence #8.5

Motion of m_2 relative to m_1 (planet)

$$\frac{d^2\mathbf{r}}{dt^2} = \underbrace{-\frac{G(m_1+m_2)}{r^3}\mathbf{r}}_{\mathbf{a}_{21}^p} \underbrace{-Gm_3\left(\frac{1}{d^3}\mathbf{d} + \frac{1}{\rho^3}\boldsymbol{\rho}\right)}_{\mathbf{a}_{21}^d}$$

Motion of m_2 relative to m_3 (sun)

$$\frac{d^2\mathbf{d}}{dt^2} = \underbrace{-\frac{G(m_2+m_3)}{d^3}\mathbf{d}}_{\mathbf{a}_{23}^p} \underbrace{-Gm_1\left(\frac{1}{r^3}\mathbf{r} - \frac{1}{\rho^3}\boldsymbol{\rho}\right)}_{\mathbf{a}_{23}^d}$$

Primary accelerations

$$\mathbf{a}_{21}^p = -\frac{G(m_1+m_2)}{r^3}\mathbf{r} = -\frac{G(m_1+m_2)}{r^2}\,\mathbf{i}_r \qquad a_{21}^p = \frac{G(m_1+m_2)}{r^2}$$

$$\mathbf{a}_{23}^p = -\frac{G(m_2+m_3)}{d^3}\mathbf{d} = -\frac{G(m_2+m_3)}{d^2}\,\mathbf{i}_d \qquad a_{23}^p = \frac{G(m_2+m_3)}{r^2 - 2r\rho\cos\alpha + \rho^2}$$

Disturbing accelerations

$$\mathbf{a}_{23}^d = -Gm_1\left(\frac{1}{r^3}\mathbf{r} - \frac{1}{\rho^3}\boldsymbol{\rho}\right) = \frac{Gm_1}{r^2}\left(\frac{r^2}{\rho^2}\,\mathbf{i}_\rho - \mathbf{i}_r\right) = \frac{Gm_1}{r^2}(x^2\,\mathbf{i}_\rho - \mathbf{i}_r)$$

$$\mathbf{a}_{21}^d = -Gm_3\left(\frac{1}{d^3}\mathbf{d} + \frac{1}{\rho^3}\boldsymbol{\rho}\right) = \left[\frac{\partial R_3}{\partial \mathbf{r}}\right]^{\mathrm{T}} = \underbrace{\frac{Gm_3}{\rho^2}\sum_{k=1}^{\infty} x^k[P_{k+1}'(\nu)\,\mathbf{i}_\rho - P_k'(\nu)\,\mathbf{i}_r]}_{\text{From Eq. (8.72)}}$$

$$\approx \frac{Gm_3}{\rho^2}x[P_2'(\nu)\,\mathbf{i}_\rho - P_1'(\nu)\,\mathbf{i}_r] = \frac{Gm_3}{\rho^2}x(3\nu\,\mathbf{i}_\rho - \mathbf{i}_r)$$

$$a_{23}^d = \frac{Gm_1}{r^2}|x^2\,\mathbf{i}_\rho - \mathbf{i}_r| = \frac{Gm_1}{r^2}\sqrt{x^4 - 2x^2\cos\alpha + 1} = \frac{Gm_1}{r^2}\sqrt{1 - 2\nu x^2 + x^4} \approx \frac{Gm_1}{r^2}$$

$$a_{21}^d = \frac{Gm_3}{\rho^2}x|3\nu\,\mathbf{i}_\rho - \mathbf{i}_r| = \frac{Gm_3}{\rho^2}x\sqrt{9\nu^2 - 6\nu\cos\alpha + 1} = \frac{Gm_3}{\rho^2}x\sqrt{1 + 3\nu^2}$$

Determine the ratios

$$\frac{a_{23}^d}{a_{23}^p} \approx \frac{Gm_1}{r^2} \times \frac{r^2 - 2r\nu\rho + \rho^2}{G(m_2+m_3)} = \frac{m_1}{m_2+m_3} \times \frac{1}{x^2} \times \underbrace{(1 - 2\nu x + x^2)}_{\approx 1}$$

$$\frac{a_{21}^d}{a_{21}^p} \approx \frac{Gm_3}{\rho^2} x\sqrt{1+3\nu^2} \times \frac{r^2}{G(m_1+m_2)} = \frac{m_3}{m_1+m_2} \times x\sqrt{1+3\nu^2} \times x^2$$

Set the ratios equal

$$\frac{a_{21}^d}{a_{21}^p} = \frac{a_{23}^d}{a_{23}^p} \quad \Longrightarrow \quad \frac{m_3}{m_1+m_2} \times \sqrt{1+3\nu^2} \times x^5 = \frac{m_1}{m_2+m_3} \times (1 - 2\nu x + x^2)$$

$$x^5 = \frac{m_1(m_1+m_2)}{m_3(m_2+m_3)} \times \frac{1}{\sqrt{1+3\nu^2}}$$

Finally, $\quad 1 \leq (1+3\nu^2)^{\frac{1}{10}} < 1.15 \quad$ and for $\quad m_2 \ll m_1 \quad$ and $\quad m_2 \ll m_3$

$$x = \boxed{\frac{r}{\rho} \approx \left(\frac{m_1}{m_3}\right)^{\frac{2}{5}}}$$

$$\text{Radius of Sphere of Influence in miles} = \left(\frac{m_P}{m_S}\right)^{\frac{2}{5}} \times a_P \text{ in miles}$$

where m_P is the mass of the Planet, a_P is the semimajor axis of the Planet and m_S is the mass of the Sun.