16.346 Astrodynamics
Fall 2008

## Lagrange's Variational Methods for Linear Equations

Consider the equation

$$\frac{d^2y}{dt^2} + y = \sec t \quad \Longrightarrow \quad \begin{array}{c} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} + y_1 = \sec t \end{array} \quad \Longrightarrow \quad \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sec t \end{bmatrix}$$

which is equivalent to

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}\mathbf{y} + \mathbf{g}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 0 \\ \sec t \end{bmatrix}$$

Now the Wronskian matrix $\mathbf{W}$

$$\mathbf{W} = \begin{bmatrix} \sin t & -\cos t \\ \cos t & \sin t \end{bmatrix} \quad \text{satisfies} \quad \frac{d\mathbf{W}}{dt} = \mathbf{F}\mathbf{W}$$

and the solution of the homogeneous equation is

$$\mathbf{y}_h = \mathbf{W}\mathbf{c} \quad \text{where} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We now seek a solution of the general equation

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}\mathbf{y} + \mathbf{g} \quad \text{of the form} \quad \mathbf{y} = \mathbf{W}\mathbf{c}(t)$$

Substitute and obtain

$$\frac{d\mathbf{W}}{dt}\mathbf{c} + \mathbf{W}\frac{d\mathbf{c}}{dt} = \mathbf{F}\mathbf{W}\mathbf{c} + \mathbf{g} \quad \text{which reduces to} \quad \mathbf{W}\frac{d\mathbf{c}}{dt} = \mathbf{g}$$

Hence

$$\frac{d\mathbf{c}}{dt} = \mathbf{W}^{-1}\mathbf{g}$$

which is solved by quadratures to obtain

$$\frac{d\mathbf{c}}{dt} = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}\begin{bmatrix} 0 \\ \sec t \end{bmatrix} \quad \text{or} \quad \frac{dc_1}{dt} = 1 \quad \text{and} \quad \frac{dc_2}{dt} = \tan t$$

Hence

$$c_1(t) = t + c_1 \quad \text{and} \quad c_2(t) = \log(\sec t + \tan t) + c_2$$

so that the general solution is simply

$$\mathbf{y} = \mathbf{W}\mathbf{c}(t) \quad \text{or} \quad y(t) = c_1(t)\sin t - c_2(t)\cos t$$

## Derivation of the Variational Equations

$$\frac{d}{dt}\begin{bmatrix}\mathbf{r}\\ \mathbf{v}\end{bmatrix} = \begin{bmatrix}\mathbf{O} & \mathbf{I}\\ -\frac{\mu}{r^3}\mathbf{I} & \mathbf{O}\end{bmatrix}\begin{bmatrix}\mathbf{r}\\ \mathbf{v}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{a}_d\end{bmatrix} \quad \Longleftrightarrow \quad \frac{d\mathbf{s}}{dt} = \mathbf{F}\,\mathbf{s} + \boldsymbol{\eta}$$

where

$$\mathbf{s} = \mathbf{s}(t,\boldsymbol{\alpha}) = \begin{bmatrix}\mathbf{r}(t,\boldsymbol{\alpha})\\ \mathbf{v}(t,\boldsymbol{\alpha})\end{bmatrix}$$

**Two-Body Motion:** $\dfrac{\partial\mathbf{s}}{\partial t} = \mathbf{F}\,\mathbf{s}$ **Disturbed Motion:** $\dfrac{d\mathbf{s}}{dt} = \mathbf{F}\,\mathbf{s} + \boldsymbol{\eta}$

Seek solutions of the form $\mathbf{s} = \mathbf{s}(t,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)$

where, for example, $\boldsymbol{\alpha}^{\mathrm{T}} = [\,\Omega \quad i \quad \omega \quad a \quad e \quad \lambda = -n\tau\,]$ and $\boldsymbol{\eta} = \begin{bmatrix}\mathbf{0}\\ \mathbf{a}_d\end{bmatrix}$

Differentiate

$$\frac{d\mathbf{s}}{dt} = \frac{\partial\mathbf{s}}{\partial t} + \frac{\partial\mathbf{s}}{\partial\boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} = \mathbf{F}\,\mathbf{s} + \boldsymbol{\eta} \quad \Longrightarrow \quad \frac{\partial\mathbf{s}}{\partial\boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} = \boldsymbol{\eta}$$

Since

$$\frac{\partial\mathbf{s}}{\partial\boldsymbol{\alpha}}\frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{s}} = \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{s}}\frac{\partial\mathbf{s}}{\partial\boldsymbol{\alpha}} = \mathbf{I}$$

then

$$\frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{s}}\frac{\partial\mathbf{s}}{\partial\boldsymbol{\alpha}}\frac{d\boldsymbol{\alpha}}{dt} = \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{s}}\boldsymbol{\eta} = \begin{bmatrix}\frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{r}} & \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{v}}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{a}_d\end{bmatrix} = \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{v}}\mathbf{a}_d$$

so that

$$\boxed{\frac{d\boldsymbol{\alpha}}{dt} = \frac{\partial\boldsymbol{\alpha}}{\partial\mathbf{v}}\mathbf{a}_d}$$

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$$\begin{aligned} x &= f(\xi,\eta)\\ y &= g(\xi,\eta)\end{aligned} \qquad \begin{bmatrix}dx\\ dy\end{bmatrix} = \begin{bmatrix}f_\xi & f_\eta\\ g_\xi & g_\eta\end{bmatrix}\begin{bmatrix}d\xi\\ d\eta\end{bmatrix} \qquad \begin{aligned}\xi &= F(x,y)\\ \eta &= G(x,y)\end{aligned} \qquad \begin{bmatrix}d\xi\\ d\eta\end{bmatrix} = \begin{bmatrix}F_x & F_y\\ G_x & G_y\end{bmatrix}\begin{bmatrix}dx\\ dy\end{bmatrix}$$

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## Variation of the Classical Elements

$$\mu\left(\frac{2}{r} - \frac{1}{a}\right) = v^2 = \mathbf{v}\cdot\mathbf{v} = \mathbf{v}^{\mathrm{T}}\mathbf{v} \quad \Longrightarrow \quad \frac{\mu}{a^2}\frac{\partial a}{\partial\mathbf{v}} = 2\mathbf{v}^{\mathrm{T}} \quad \Longrightarrow \quad \boxed{\frac{da}{dt} = \frac{2a^2}{\mu}\mathbf{v}\cdot\mathbf{a}_d}$$

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} = \mathbf{S}_r\mathbf{v} \quad \Longrightarrow \quad \frac{\partial\mathbf{h}}{\partial\mathbf{v}} = \mathbf{S}_r\frac{\partial\mathbf{v}}{\partial\mathbf{v}} = \mathbf{S}_r\mathbf{I} = \mathbf{S}_r \quad \Longrightarrow \quad \boxed{\frac{d\mathbf{h}}{dt} = \mathbf{r}\times\mathbf{a}_d}$$

$$h^2 = \mathbf{h}^{\mathrm{T}}\mathbf{h} \quad \Longrightarrow \quad 2h\frac{\partial h}{\partial \mathbf{v}} = 2\mathbf{h}^{\mathrm{T}}\mathbf{S}_r \quad \Longrightarrow \quad \boxed{\frac{dh}{dt} = \mathbf{i}_h \cdot \mathbf{r} \times \mathbf{a}_d = \mathbf{i}_h \times \mathbf{r} \cdot \mathbf{a}_d = r\,\mathbf{i}_\theta \cdot \mathbf{a}_d}$$

or, alternately,

$$h^2 = (\mathbf{r} \times \mathbf{v}) \cdot (\mathbf{r} \times \mathbf{v}) = \mathbf{r}^{\mathrm{T}}\mathbf{r}\mathbf{v}^{\mathrm{T}}\mathbf{v} - \mathbf{r}^{\mathrm{T}}\mathbf{v}\mathbf{r}^{\mathrm{T}}\mathbf{v} \quad \Longrightarrow \quad \boxed{\frac{dh}{dt} = \frac{1}{h}\mathbf{r}^{\mathrm{T}}(\mathbf{r}\mathbf{v}^{\mathrm{T}} - \mathbf{v}\mathbf{r}^{\mathrm{T}})\mathbf{a}_d}$$

$$p = \frac{h^2}{\mu} = a(1-e^2) \quad \Longrightarrow \quad \boxed{2\mu a e\frac{de}{dt} = \mu(1-e^2)\frac{da}{dt} - 2h\frac{dh}{dt}}$$

$$\boxed{\mu\frac{d\mathbf{e}}{dt} = \mathbf{a}_d \times (\mathbf{r} \times \mathbf{v}) + (\mathbf{a}_d \times \mathbf{r}) \times \mathbf{v}}$$

## Variation of $i$ and $\Omega$

From Page 84 in the textbook

$$\mathbf{h} = h\,\mathbf{i}_h = h(\sin\Omega\sin i\,\mathbf{i}_x - \cos\Omega\sin i\,\mathbf{i}_y + \cos i\,\mathbf{i}_z) \tag{2.6}$$

Then

$$\frac{d\mathbf{h}}{dt} = h\sin i\frac{d\Omega}{dt}\,\mathbf{i}_n - h\frac{di}{dt}\,\mathbf{i}_m + \frac{dh}{dt}\,\mathbf{i}_h$$

where

$$\mathbf{i}_n = \cos\Omega\,\mathbf{i}_x + \sin\Omega\,\mathbf{i}_y \tag{2.5}$$

$$\mathbf{i}_m = \mathbf{i}_h \times \mathbf{i}_n = -\sin\Omega\cos i\,\mathbf{i}_x + \cos\Omega\cos i\,\mathbf{i}_y + \sin i\,\mathbf{i}_z \tag{2.8}$$

Hence

$$\boxed{\begin{aligned} \frac{d\Omega}{dt} &= \frac{1}{h\sin i}\mathbf{i}_n \times \mathbf{r} \cdot \mathbf{a}_d = \frac{r\sin\theta}{h\sin i}\mathbf{i}_h \cdot \mathbf{a}_d \\ \frac{di}{dt} &= -\frac{1}{h}\mathbf{i}_m \times \mathbf{r} \cdot \mathbf{a}_d = \frac{r\cos\theta}{h}\mathbf{i}_h \cdot \mathbf{a}_d \end{aligned}}$$

where $\theta = \omega + f$ is the argument of latitude.

## Variation of the true anomaly $f$

$$r(1 + e\cos f) = \frac{h^2}{\mu} \quad \Longrightarrow \quad re\sin f\frac{\partial f}{\partial \mathbf{v}} = r\cos f\frac{\partial e}{\partial \mathbf{v}} - \frac{2h}{\mu}\frac{\partial h}{\partial \mathbf{v}}$$

From Eq. (3.29) $\quad \frac{\mu}{h}re\sin f = \mathbf{r} \cdot \mathbf{v} \quad \Longrightarrow \quad re\cos f\frac{\partial f}{\partial \mathbf{v}} = -r\sin f\frac{\partial e}{\partial \mathbf{v}} + \frac{\mathbf{r} \cdot \mathbf{v}}{\mu}\frac{\partial h}{\partial \mathbf{v}} + \frac{h}{\mu}\mathbf{r}^{\mathrm{T}}$

Multiply the first by $\cos f$, the second by $\sin f$ and add to obtain

$$reh\frac{\partial f}{\partial \mathbf{v}} = (p\cos f)\,\mathbf{r}^{\mathrm{T}} - (p + r)\sin f\frac{\partial h}{\partial \mathbf{v}} \quad \text{to be used in} \quad \boxed{\frac{df}{dt} = \frac{h}{r^2} + \frac{\partial f}{\partial \mathbf{v}}\mathbf{a}_d}$$

## Variation of $\omega$

$$\mathbf{i}_n = \cos\Omega\,\mathbf{i}_x + \sin\Omega\,\mathbf{i}_y \quad \Longrightarrow \quad \cos\theta = \mathbf{i}_n \cdot \mathbf{i}_r = \cos\Omega\,(\mathbf{i}_x \cdot \mathbf{i}_r) + \sin\Omega\,(\mathbf{i}_y \cdot \mathbf{i}_r)$$

Then

$$-\sin\theta\,\frac{\partial\theta}{\partial\mathbf{v}} = [-\sin\Omega\,(\mathbf{i}_x \cdot \mathbf{i}_r) + \cos\Omega\,(\mathbf{i}_y \cdot \mathbf{i}_r)]\frac{\partial\Omega}{\partial\mathbf{v}} \quad \Longrightarrow \quad \frac{\partial\theta}{\partial\mathbf{v}} = -\cos i\,\frac{\partial\Omega}{\partial\mathbf{v}}$$

since

$$\mathbf{i}_x \cdot \mathbf{i}_r = \cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i$$
$$\mathbf{i}_y \cdot \mathbf{i}_r = \sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i$$

This gives the perturbative derivative of $\theta$, i.e., the change in $\theta$ due to the change in $\mathbf{i}_n$ from which the angle $\theta$ is measured. The total time rate of change of $\theta$ is the sum

$$\frac{d\theta}{dt} = \frac{\partial\theta}{\partial t} + \frac{\partial\theta}{\partial\mathbf{v}}\mathbf{a}_d = \frac{h}{r^2} - \cos i\frac{d\Omega}{dt}$$

Since $\theta = \omega + f$, then

$$\boxed{\frac{d\omega}{dt} = -\frac{\partial f}{\partial\mathbf{v}}\,\mathbf{a}_d - \cos i\,\frac{d\Omega}{dt}}$$

## Gauss' form of Lagrange's variational equations in polar coordinates

$$\begin{aligned}
\frac{d\Omega}{dt} &= \frac{r\sin\theta}{h\sin i}a_{dh}\\
\frac{di}{dt} &= \frac{r\cos\theta}{h}a_{dh}\\
\frac{d\omega}{dt} &= \frac{1}{he}[-p\cos f\,a_{dr} + (p+r)\sin f\,a_{d\theta}] - \frac{r\sin\theta\cos i}{h\sin i}a_{dh}\\
\frac{da}{dt} &= \frac{2a^2}{h}\left(e\sin f\,a_{dr} + \frac{p}{r}a_{d\theta}\right)\\
\frac{de}{dt} &= \frac{1}{h}\{p\sin f\,a_{dr} + [(p+r)\cos f + re]a_{d\theta}\}\\
\frac{df}{dt} &= \frac{h}{r^2} + \frac{1}{eh}[p\cos f\,a_{dr} - (p+r)\sin f\,a_{d\theta}]
\end{aligned}$$

## Gauss' form of the variational equations in tangential-normal coordinates

$$\begin{aligned}
\frac{d\omega}{dt} &= \frac{1}{ev}\left[2\sin f\,a_{dt} + \left(2e + \frac{r}{a}\cos f\right)a_{dn}\right] - \frac{r\sin\theta\cos i}{h\sin i}a_{dh}\\
\frac{da}{dt} &= \frac{2a^2v}{\mu}a_{dt}\\
\frac{de}{dt} &= \frac{1}{v}\left[2(e+\cos f)a_{dt} - \frac{r}{a}\sin f\,a_{dn}\right]\\
\frac{df}{dt} &= \frac{h}{r^2} - \frac{1}{ev}\left[2\sin f\,a_{dt} + \left(2e + \frac{r}{a}\cos f\right)a_{dn}\right]
\end{aligned}$$