16.346 Astrodynamics
Fall 2008

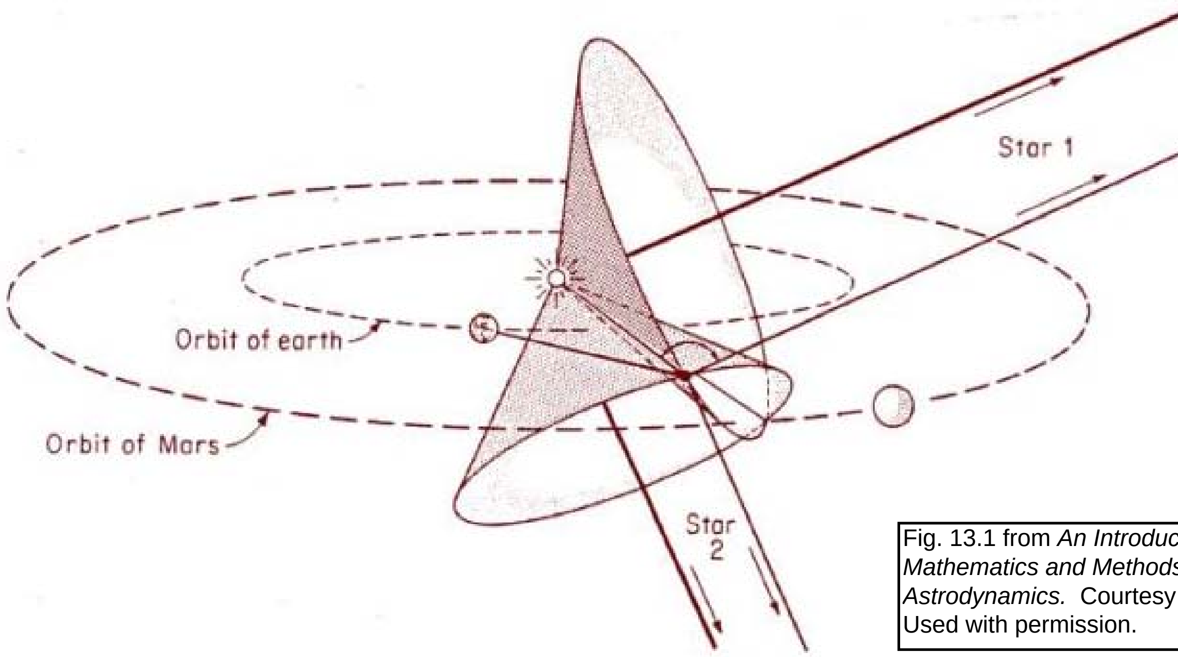

Fig. 13.1 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

The Line of Position

$$\begin{aligned} \mathbf{i}_r \cdot \mathbf{i}_{\star 1} &= -\cos A_1 \\ \mathbf{i}_r \cdot \mathbf{i}_{\star 2} &= -\cos A_2 \end{aligned} \quad \Longrightarrow \quad \mathbf{i}_r = \alpha\,\mathbf{i}_{\star 1} + \beta\,\mathbf{i}_{\star 2} + \gamma\,\mathbf{i}_{\star 1} \times \mathbf{i}_{\star 2}$$

where

$$\begin{aligned} \alpha \sin^2\varphi &= \cos A_2 \cos\varphi - \cos A_1 \\ \beta \sin^2\varphi &= \cos A_1 \cos\varphi - \cos A_2 \\ \gamma^2 \sin^2\varphi &= 1 + \alpha \cos A_1 + \beta \cos A_2 \end{aligned}$$

and $\cos\varphi = \mathbf{i}_{\star 1} \cdot \mathbf{i}_{\star 2}$

The Position Fix

$$\begin{aligned} \mathbf{i}_r \cdot \mathbf{i}_{\star 1} &= -\cos A_1 \\ \mathbf{i}_r \cdot \mathbf{i}_{\star 2} &= -\cos A_2 \\ \mathbf{i}_r \cdot \mathbf{r}_p &= r - |\mathbf{r}_p - \mathbf{r}| \cos A_3 \end{aligned}$$

where $\mathbf{r}_p$ is the position vector of a planet or other near object.

Henceforth, we will linearize the measurements so that we can deal with a set of redundant measurements using Gauss's Method of Least Squares.

Determining the Measurement Geometry Vector #13.2

For an arbitrary angle $A(\mathbf{r})$, we calculate the measurement geometry vector $\mathbf{h}$ from the Taylor Series expansion about a reference position $\mathbf{r}_0$ and discarding all terms of higher order in δr:

$$A(\mathbf{r}) = A(\mathbf{r}_0) + \left.\frac{\partial A}{\partial \mathbf{r}}\right|_{\mathbf{r}=\mathbf{r}_0} \delta\mathbf{r} + \cdots = A_0 + \mathbf{h}^{\mathsf{T}}\delta\mathbf{r} + \cdots$$

Hence

$$\boxed{\delta A = h^{\mathsf{T}}\delta\mathbf{r} \quad \text{where} \quad \mathbf{h}^{\mathsf{T}} = \left.\frac{\partial A}{\partial \mathbf{r}}\right|_{\mathbf{r}=\mathbf{r}_0}}$$

Measuring the Angle between a Near Object and a Star

The angle between the line-of-sight to a near object, e.g., the sun or a planet, and the line-of-sight to a distant star is defined by

$$r\cos A = -\mathbf{i}_\star^{\mathsf{T}}\mathbf{r}$$

from which

$$\frac{\partial r}{\partial \mathbf{r}}\cos A - r\sin A\,\frac{\partial A}{\partial \mathbf{r}} = -\mathbf{i}_\star^{\mathsf{T}}$$

The derivative of the scalar r with respect to the vector $\mathbf{r}$ is obtained from

$$r^2 = \mathbf{r}\cdot\mathbf{r} = \mathbf{r}^{\mathsf{T}}\mathbf{r} \quad\Longrightarrow\quad 2r\frac{\partial r}{\partial \mathbf{r}} = 2\mathbf{r}^{\mathsf{T}}\frac{\partial \mathbf{r}}{\partial \mathbf{r}} = 2\mathbf{r}^{\mathsf{T}}\mathbf{I} \quad\Longrightarrow\quad \frac{\partial r}{\partial \mathbf{r}} = \frac{1}{r}\mathbf{r}^{\mathsf{T}} = \mathbf{i}_r^{\mathsf{T}}$$

Therefore,

$$\boxed{\mathbf{h} = \frac{1}{r\sin A}(\cos A\,\mathbf{i}_r + \mathbf{i}_\star)} \quad \text{or} \quad \boxed{\mathbf{h} = \frac{1}{r}\mathbf{i}_n}$$

The vector $\mathbf{i}_n$ is a unit vector in the plane of the measurement and perpendicular to the line-of-sight to the near object.

Measuring the Angle between Two Near Objects

The angle between the two position vectors $\mathbf{r}$ and $\mathbf{d}$ produces the measurement equation

$$\mathbf{d}^{\mathsf{T}}\mathbf{r} = dr\cos A$$

Since $\mathbf{r} - \mathbf{d} = \text{constant}$, then $\delta\mathbf{d} = \delta\mathbf{r}$. Again, from $d^2 = \mathbf{d}^{\mathsf{T}}\mathbf{d}$, we have

$$2d\frac{\partial d}{\partial \mathbf{r}} = 2\mathbf{d}^{\mathsf{T}}\mathbf{I} \quad \text{or} \quad \frac{\partial d}{\partial \mathbf{r}} = \mathbf{i}_d^{\mathsf{T}}$$

Hence:

$$\boxed{\mathbf{h} = -\frac{1}{r\sin A}(\mathbf{i}_d - \cos A\,\mathbf{i}_r) - \frac{1}{d\sin A}(\mathbf{i}_r - \cos A\,\mathbf{i}_d)} \quad \text{or} \quad \boxed{\mathbf{h} = \frac{1}{r}\mathbf{i}_n + \frac{1}{d}\mathbf{i}_m}$$

Both $\mathbf{i}_n$ and $\mathbf{i}_m$ are unit vectors in the plane determined by the spacecraft and the two near bodies.

The Measurement Geometry Matrix

For several measurements we define

$$\mathbf{H} = [\mathbf{h}_1 \quad \mathbf{h}_2 \ \cdots \ \mathbf{h}_n] \qquad \text{so that} \qquad \delta\mathbf{q} = \mathbf{H}^{\mathsf{T}} \delta\mathbf{r}$$

where

$$\delta\mathbf{q} = \begin{bmatrix} \delta q_1 \\ \delta q_2 \\ \vdots \\ \delta q_n \end{bmatrix} \qquad \text{and} \qquad \partial\mathbf{r} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix}$$

Gauss' Method of Least Squares #13.5

Given m_{ij} and c_i: To determine x_i so that

$$\sum_{j=1}^{n} m_{ij} x_j = c_i \quad \text{where} \quad i = 1, 2, \ldots, N > n$$

is "as nearly satisfied as possible."

- Define: Residuals $e_i = \sum_{j=1}^{n} m_{ij} x_j - c_i$
- Choose: Weighting factor $w_i > 0$ for i^{th} residual
- Determine: $x_1, x_2, \ldots, x_n$ so that $w_1 e_1^2 + w_2 e_2^2 + \cdots + w_N e_N^2$ is a minimum.

Solution of Least Squares Problem

- Vector of residuals: $\mathbf{e} = \mathbf{M}\mathbf{x} - \mathbf{c}$
- Weighting matrix: $\mathbf{W} = \begin{bmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_N \end{bmatrix} = \mathbf{W}^{\mathsf{T}}$
- Weighted squares:

$$\mathbf{e}^{\mathsf{T}}\mathbf{W}\mathbf{e} = (\mathbf{x}^{\mathsf{T}}\mathbf{M}^{\mathsf{T}} - \mathbf{c}^{\mathsf{T}})\mathbf{W}(\mathbf{M}\mathbf{x} - \mathbf{c}) = \mathbf{x}^{\mathsf{T}}\mathbf{M}^{\mathsf{T}}\mathbf{W}\mathbf{M}\mathbf{x} - \mathbf{c}^{\mathsf{T}}\mathbf{W}\mathbf{M}\mathbf{x} - \mathbf{x}^{\mathsf{T}}\mathbf{M}^{\mathsf{T}}\mathbf{W}\mathbf{c} + \mathbf{c}^{\mathsf{T}}\mathbf{W}\mathbf{c}$$

- Least value of the weighted squares:

$$\frac{\partial}{\partial \mathbf{x}}(\mathbf{e}^{\mathsf{T}}\mathbf{W}\mathbf{e}) = 2\mathbf{x}^{\mathsf{T}}\mathbf{M}^{\mathsf{T}}\mathbf{W}\mathbf{M} - 2\mathbf{c}^{\mathsf{T}}\mathbf{W}\mathbf{M} = \mathbf{0}^{\mathsf{T}} \quad \text{or} \quad \mathbf{M}^{\mathsf{T}}\mathbf{W}\mathbf{M}\mathbf{x} = \mathbf{M}^{\mathsf{T}}\mathbf{W}\mathbf{c}$$

$$\boxed{\mathbf{x} = (\mathbf{M}^{\mathsf{T}}\mathbf{W}\mathbf{M})^{-1}\mathbf{M}^{\mathsf{T}}\mathbf{W}\mathbf{c}}$$

Note: If $y = \mathbf{x}^{\mathsf{T}}\mathbf{B}\mathbf{x} = (\mathbf{x}^{\mathsf{T}}\mathbf{B}\mathbf{x})^{\mathsf{T}} = \mathbf{x}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\mathbf{x}$ then $\dfrac{\partial y}{\partial \mathbf{x}} = \mathbf{x}^{\mathsf{T}}\mathbf{B}\mathbf{I} + \mathbf{x}^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\mathbf{I}$

Application of Gauss' Method of Least Squares to Space Navigation

$$\mathbf{x} = (\mathbf{M}^{\mathrm{T}}\mathbf{W}\mathbf{M})^{-1}\mathbf{M}^{\mathrm{T}}\mathbf{W}\mathbf{c}$$

In our notation

$$\mathbf{x} = \delta\widehat{\mathbf{r}} \qquad \mathbf{M}^{\mathrm{T}} = \mathbf{H} \qquad \mathbf{W} = \mathbf{A}^{-1} \qquad w_i = \frac{1}{\sigma_i^2} \qquad \mathbf{c} = \delta\widetilde{\mathbf{q}}$$

so that

$$\boxed{\delta\widehat{\mathbf{r}} = \mathbf{F}\,\delta\widetilde{\mathbf{q}}}$$

$\mathbf{F}$ is called the Estimator Matrix where

$$\boxed{\mathbf{F} = \mathbf{P}\mathbf{H}\mathbf{A}^{-1}} \qquad \text{and} \qquad \boxed{\mathbf{P} = (\mathbf{H}\mathbf{A}^{-1}\mathbf{H}^{\mathrm{T}})^{-1}}$$

The deviation from the reference position is an estimate, denoted by the "hat" over the position vector

$$\boxed{\delta\widehat{\mathbf{r}} = \delta\mathbf{r} + \boldsymbol{\epsilon}}$$

and is the sum of the actual deviation and the error in the estimate. Similarly,

$$\boxed{\delta\widetilde{\mathbf{q}} = \delta\mathbf{q} + \boldsymbol{\alpha}}$$

where $\boldsymbol{\alpha}$ is the vector error in the determination of the quantities measured. (The vector $\delta\mathbf{q}$ is the actual deviation in those quantities from their reference values.)

The Information Matrix

The matrix $\mathbf{P}^{-1}$ is called the Information Matrix because of the property

$$\mathbf{H}\mathbf{A}^{-1}\mathbf{H}^{\mathrm{T}} = \frac{\mathbf{h}_1\mathbf{h}_1^{\mathrm{T}}}{\sigma_1^2} + \frac{\mathbf{h}_2\mathbf{h}_2^{\mathrm{T}}}{\sigma_2^2} + \cdots$$

Each new measurement adds a new term to the series and each term contains all the information about the new measurement.

$$\boxed{\mathbf{P}^{-1} = \mathbf{H}\mathbf{A}^{-1}\mathbf{H}^{\mathrm{T}} = \sum_{i=1}^{N} \frac{\mathbf{h}_i\mathbf{h}_i^{\mathrm{T}}}{\sigma_i^2}}$$

The factor σ_i^2 is called the variance. The larger the i^{th} variance the less weight is given to the i^{th} measurement.