16.346 Astrodynamics
Fall 2008

Powered Flight Maneuver Equations

Required velocity: $\mathbf{v}_r(t,\mathbf{r})$ Velocity-to-be-gained: $\mathbf{v}_g(t)$ Vehicle velocity: $\mathbf{v}(t)$

$$\mathbf{v} + \mathbf{v}_g = \mathbf{v}_r$$

$$\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}_g}{dt} = \frac{d\mathbf{v}_r}{dt} = \frac{\partial \mathbf{v}_r}{\partial t} + \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}\frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{v}_r}{\partial t} + \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}\mathbf{v} = \frac{\partial \mathbf{v}_r}{\partial t} + \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}\mathbf{v}_r - \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}\mathbf{v}_g$$

$$\mathbf{g}(\mathbf{r}) + \mathbf{a}_T + \frac{d\mathbf{v}_g}{dt} = \mathbf{g}(\mathbf{r}) - \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}\mathbf{v}_g$$

$$\frac{d\mathbf{v}_g}{dt} = -\frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}\mathbf{v}_g - \mathbf{a}_T \quad \Longrightarrow \quad \boxed{\frac{d\mathbf{v}_g}{dt} = -\mathbf{C}^\star\mathbf{v}_g - \mathbf{a}_T} \quad \text{where} \quad \boxed{\mathbf{C}^\star = \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}}}$$

This is the fundamental equation for the velocity-to-be-gained.

Constant Gravity Field Example

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \qquad \mathbf{r}(t_0) = \mathbf{r}_0$$

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} \qquad \mathbf{v}(t_0) = \mathbf{v}_0$$

$$\Longrightarrow \quad \mathbf{r}(t) = \mathbf{r}_0 + (t-t_0)\mathbf{v}_0 + \tfrac{1}{2}(t-t_0)^2\mathbf{g}$$

or, with change of notation,

$$\mathbf{r}(t_1) = \mathbf{r}(t) + (t_1 - t)\mathbf{v}_r(t) + \tfrac{1}{2}(t_1-t)^2\mathbf{g}$$

Hence

$$\mathbf{v}_r(t) = \frac{1}{t_1 - t}[\mathbf{r}_1 - \mathbf{r}(t) - \tfrac{1}{2}(t_1-t)^2\mathbf{g}]$$

$$\mathbf{C}^\star(t) = \frac{\partial \mathbf{v}_r}{\partial \mathbf{r}} = -\frac{1}{t_1 - t}\mathbf{I}$$

$$\Longrightarrow \quad \frac{d\mathbf{v}_g}{dt} = \frac{1}{t_1 - t}\mathbf{v}_g - \mathbf{a}_T$$

To optimize the control of the thrust direction for the purpose of minimizing the fuel consumption, we convert the equation for the velocity-to-be-gained vector to scalar form by multiplying both sides sides by $\mathbf{v}_g$

$$2\mathbf{v}_g \cdot \frac{d\mathbf{v}_g}{dt} = \frac{d}{dt}(\mathbf{v}_g \cdot \mathbf{v}_g) = \frac{d}{dt}v_g^2 = \frac{2}{t_1 - t}v_g^2 - 2\mathbf{a}_T \cdot \mathbf{v}_g$$

$$(t_1 - t)\frac{d}{dt}v_g^2 = 2v_g^2 - 2(t_1 - t)\mathbf{a}_T \cdot \mathbf{v}_g$$

Next, integrate by parts from the current time t to the time of engine cut-off t_{co}

$$(t_1 - t)v_g^2(t)\Big|_t^{t_{co}} + \int_t^{t_{co}} v_g^2(t)\,dt = \int_t^{t_{co}} [2v_g^2(t) - 2(t_1-t)\mathbf{a}_T \cdot \mathbf{v}_g]\,dt$$

$$\int_t^{t_{co}} [2(t_1-t)\mathbf{a}_T \cdot \mathbf{v}_g - v_g^2]\,dt = (t_1 - t)v_g^2(t) = \text{ constant at the time } t$$

To minimize the time interval $t_{co} - t$, choose $\boxed{\mathbf{a}_T \parallel \mathbf{v}_g}$ or equivalently, $\boxed{\frac{d\mathbf{v}_g}{dt} \times \mathbf{v}_g = \mathbf{0}}$

to maximize the integrand—the latter is referred to as **Cross-Product Steering**

Vehicle Orientation Prior to Cross-Product Steering

Define a vector $\mathbf{p}(t) = -\mathbf{C}^\star(t)\mathbf{v}_g(t) = \dfrac{d\mathbf{v}_r}{dt} - \mathbf{g}(\mathbf{r})$ so that $\dfrac{d\mathbf{v}_g}{dt} = \mathbf{p}(t) - \mathbf{a}_T$

Then, to determine the direction of the thrust acceleration vector $\mathbf{a}_T$

$$\frac{d\mathbf{v}_g}{dt} \times \mathbf{v}_g = \mathbf{0} \quad \Longrightarrow \quad \mathbf{a}_T \times \mathbf{v}_g = \mathbf{p} \times \mathbf{v}_g$$

$$(\mathbf{a}_T \times \mathbf{v}_g) \times \mathbf{v}_g = (\mathbf{p} \times \mathbf{v}_g) \times \mathbf{v}_g$$

$$(\mathbf{a}_T \cdot \mathbf{v}_g)\mathbf{v}_g - v_g^2 \mathbf{a}_T = (\mathbf{p} \cdot \mathbf{v}_g)\mathbf{v}_g - v_g^2 \mathbf{p}$$

$$\mathbf{a}_T = \mathbf{p} + (q - \mathbf{i}_{v_g} \cdot \mathbf{p})\,\mathbf{i}_{v_g} \quad \text{where} \quad q = \mathbf{a}_T \cdot \mathbf{i}_{v_g}$$

To obtain q: $\quad \mathbf{a}_T \cdot \mathbf{a}_T = a_T^2 = [\mathbf{p} + (q - \mathbf{i}_{v_g} \cdot \mathbf{p})\,\mathbf{i}_{v_g}] \cdot [\mathbf{p} + (q - \mathbf{i}_{v_g} \cdot \mathbf{p})\,\mathbf{i}_{v_g}]$

so that $\quad q = \sqrt{a_T^2 - p^2 + (\mathbf{i}_{v_g} \cdot \mathbf{p})^2}$

Clearly, cross-product steering is not possible unless $a_T^2 > p^2$.

Hyperbolic Injection Guidance

$$\mathbf{v}_r = \tfrac{1}{2} v_\infty [(D+1)\mathbf{i}_\infty + (D-1)\mathbf{i}_r]$$

where $\quad v_\infty^2 (r + \mathbf{i}_\infty^{\mathrm{T}} \mathbf{r})(D^2 - 1) = 4\mu$

To obtain the $\mathbf{C}^\star$ matrix, first calculate:

$$\frac{\partial \mathbf{v}_r}{\partial \mathbf{r}} = \tfrac{1}{2} v_\infty (\mathbf{i}_\infty + \mathbf{i}_r) \frac{\partial D}{\partial \mathbf{r}} + \tfrac{1}{2} v_\infty (D-1) \frac{\partial \mathbf{i}_r}{\partial \mathbf{r}}$$

$$\frac{\partial \mathbf{i}_r}{\partial \mathbf{r}} = \frac{\partial}{\partial \mathbf{r}} \left(\frac{\mathbf{r}}{r} \right) = \frac{1}{r} \frac{\partial \mathbf{r}}{\partial \mathbf{r}} - \frac{1}{r^2} \mathbf{r} \frac{\partial r}{\partial \mathbf{r}} = \frac{1}{r} (\mathbf{I} - \mathbf{i}_r \mathbf{i}_r^{\mathrm{T}})$$

$$\frac{\partial D}{\partial \mathbf{r}} = -v_\infty^2 \frac{(D^2 - 1)}{8\mu D} (\mathbf{i}_\infty + \mathbf{i}_r)^{\mathrm{T}}$$

Then

$$\boxed{\mathbf{C}^\star = \frac{v_\infty (D-1)}{2r} (\mathbf{I} - \mathbf{i}_r \mathbf{i}_r^{\mathrm{T}}) - \frac{v_\infty^3 (D^2 - 1)^2}{16 \mu D} (\mathbf{i}_\infty + \mathbf{i}_r)(\mathbf{i}_\infty + \mathbf{i}_r)^{\mathrm{T}}}$$

Circular Orbit Insertion Guidance

$$\mathbf{v}_r = \sqrt{\frac{\mu}{r}}\, \mathbf{i}_n \times \mathbf{i}_r = \mathbf{S}_n \mathbf{r} \sqrt{\frac{\mu}{r^3}}$$

where $\mathbf{i}_n$ and $\mathbf{S}_n$ are

$$\mathbf{i}_n = \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \qquad \text{and} \qquad \mathbf{S}_n = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

Then

$$\boxed{\mathbf{C}^\star = \sqrt{\frac{\mu}{r^3}}\, \mathbf{S}_n (\mathbf{I} - \tfrac{3}{2} \mathbf{i}_r \mathbf{i}_r^{\mathrm{T}})}$$