16.346 Astrodynamics
Fall 2008

Taylor's Series (1712) **Brook Taylor (1685–1731)**

The vector $\mathbf{r}$ can be expressed as a Taylor series about $\mathbf{r}_1$ with time interval $\tau = t_2 - t_1$:

$$\mathbf{r}_2 = \mathbf{r}_1 + \tau\mathbf{v}_1 + \frac{\tau^2}{2!}\mathbf{r}_1'' + \frac{\tau^3}{3!}\mathbf{r}_1''' + O(\tau^4)$$

Approximate Solution of the BVP using Taylor's Series

Differentiate the series twice and use the equation of motion

$$\mathbf{r}_2'' + \epsilon_2\mathbf{r}_2 = \mathbf{0} \qquad \text{where} \qquad \epsilon_2 = \frac{\mu}{r_2^3}$$

Then

$$\mathbf{r}_2'' = \mathbf{r}_1'' + \tau\mathbf{r}_1'''$$
$$-\epsilon_2\mathbf{r}_2 = -\epsilon_1\mathbf{r}_1 + \tau\mathbf{r}_1'''$$

Hence

$$\mathbf{r}_2 = \mathbf{r}_1 + \tau\mathbf{v}_1 - \frac{\tau^2}{2!}\epsilon_1\mathbf{r}_1 + \frac{\tau^2}{3!}(\epsilon_1\mathbf{r}_1 - \epsilon_2\mathbf{r}_2)$$

Solve for $\tau\mathbf{v}_1$ to obtain

$$\boxed{\mathbf{v}_1 = -\left(\frac{1}{\tau} - \frac{\epsilon_1}{3}\tau\right)\mathbf{r}_1 + \left(\frac{1}{\tau} + \frac{\epsilon_2}{6}\tau\right)\mathbf{r}_2}$$

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valid to third order in the time interval $\tau = t_2 - t_1$.

Another Method of Gibbs using Taylor Series **Pages 136–137**

$$r(t) = a_0 + a_1(t-\tau) + a_2(t-\tau)^2 + a_3(t-\tau)^3 + a_4(t-\tau)^4 + O[(t-\tau)^5]$$

where $a_0,\ a_1,\ a_2,\ldots$ are the function and its derivatives evaluated at time $t = \tau$.

- Given $r_1 = r(t_1),\ r_2 = r(t_2),\ r_3 = r(t_3)$
- To determine p we have six equations in the six unknowns $a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ p$:

$$\begin{aligned} r_1 &= a_0 - a_1\tau_3 + a_2\tau_3^2 - a_3\tau_3^3 + a_4\tau_3^4 \\ r_2 &= a_0 \\ r_3 &= a_0 + a_1\tau_1 + a_2\tau_1^2 + a_3\tau_1^3 + a_4\tau_1^4 \end{aligned} \qquad \text{where} \qquad \begin{aligned} \tau_1 &= t_3 - t_2 \\ \tau_2 &= t_3 - t_1 = \tau_1 + \tau_3 \\ \tau_3 &= t_2 - t_1 \end{aligned}$$

Also

$$\begin{aligned} \epsilon_1(p - r_1) &= 2a_2 - 6a_3\tau_3 + 12a_4\tau_3^2 \\ \epsilon_2(p - r_2) &= 2a_2 \\ \epsilon_3(p - r_3) &= 2a_2 + 6a_3\tau_1 + 12a_4\tau_1^2 \end{aligned} \qquad \text{since} \qquad \boxed{\frac{d^2r}{dt^2} = \epsilon(p - r)} \qquad \text{where} \quad \epsilon = \frac{\mu}{r^3}$$

Étienne Bezout's Theorem

Consider the system of linear algebraic system

$$
\begin{aligned}
a_1 x + a_2 y + q_1 &= 0 \\
b_1 x + b_2 y + q_2 &= 0
\end{aligned}
$$

of two equations in two unknowns x and y. If a third equation is included,

$$
\begin{aligned}
a_1 x + a_2 y + q_1 &= 0 \\
b_1 x + b_2 y + q_2 &= 0 \\
c_1 x + c_2 y + q_3 &= 0
\end{aligned}
$$

the system is now over-determined.

A necessary and sufficient condition for the system to be consistent is that

$$
\begin{vmatrix} a_1 & a_2 & q_1 \\ b_1 & b_2 & q_2 \\ c_1 & c_2 & q_3 \end{vmatrix} = 0
$$

Since we require only the parameter p, we can choose p to make the system consistent

$$
\begin{vmatrix}
1 & -\tau_3 & \tau_3^2 & -\tau_3^3 & \tau_3^4 & r_1 \\
1 & 0 & 0 & 0 & 0 & r_2 \\
1 & \tau_1 & \tau_1^2 & \tau_1^3 & \tau_1^4 & r_3 \\
0 & 0 & 2 & -6\tau_3 & 12\tau_3^2 & \epsilon_1(p - r_1) \\
0 & 0 & 2 & 0 & 0 & \epsilon_2(p - r_2) \\
0 & 0 & 2 & 6\tau_1 & 12\tau_1^2 & \epsilon_3(p - r_3)
\end{vmatrix} = 0
$$

Thus

$$
\boxed{p = \frac{r_1\tau_1(1+\epsilon_1 A_1) - r_2\tau_2(1-\epsilon_2 A_2) + r_3\tau_3(1+\epsilon_3 A_3)}{\tau_1\epsilon_1 A_1 + \tau_2\epsilon_2 A_2 + \tau_3\epsilon_3 A_3}}
$$

where $\quad 12A_1 = \tau_2\tau_3 - \tau_1^2 \qquad 12A_2 = \tau_1\tau_3 + \tau_2^2 \qquad 12A_3 = \tau_1\tau_2 - \tau_3^2$

- To determine a we again have six equations in six unknowns with the last three equations created from the relation:

$$
\boxed{\frac{d^2}{dt^2}(r^2) = 2\mu\left(\frac{1}{r} - \frac{1}{a}\right)}
$$

Problem 3–7

Then, in the same manner as before, we find

$$
\boxed{\frac{\mu}{a} = -\frac{r_1^2}{\tau_2\tau_3}(1 - 2\epsilon_1 A_1) + \frac{r_2^2}{\tau_1\tau_3}(1 + 2\epsilon_2 A_2) - \frac{r_3^2}{\tau_1\tau_2}(1 - 2\epsilon_3 A_3)}
$$

Laplace's Method (1780)

- Given the unit vectors $\mathbf{i}_{\rho_1}(t_1)$, $\mathbf{i}_{\rho_2}(t_2)$, $\mathbf{i}_{\rho_3}(t_3)$ where $\mathbf{r} = \boldsymbol{\rho} + \mathbf{d}$ and $\boldsymbol{\rho} = \rho\,\mathbf{i}_\rho$
- Determine ρ_2, $d\rho_2/dt$, $d\,\mathbf{i}_{\rho_2}/dt$ from which we obtain

$$\mathbf{r}_2 = \rho_2\,\mathbf{i}_{\rho_2} + \mathbf{d}_2$$
$$\mathbf{v}_2 = \frac{d\rho_2}{dt}\mathbf{i}_{\rho_2} + \rho_2\frac{d\,\mathbf{i}_{\rho_2}}{dt} + \frac{d\mathbf{d}_2}{dt}$$

From the second derivative of the vector $\boldsymbol{\rho} = \rho\,\mathbf{i}_\rho$

$$\frac{d^2\boldsymbol{\rho}}{dt^2} = \rho\frac{d^2\mathbf{i}_\rho}{dt^2} + 2\frac{d\rho}{dt}\frac{d\mathbf{i}_\rho}{dt} + \frac{d^2\rho}{dt^2}\mathbf{i}_\rho$$

and the equations of motion

$$\frac{d^2\boldsymbol{\rho}}{dt^2} = \frac{d^2\mathbf{r}}{dt^2} - \frac{d^2\mathbf{d}}{dt^2} = \frac{\mu}{d^3}\mathbf{d} - \frac{\mu}{r^3}\mathbf{r} = \frac{\mu}{d^3}\mathbf{d} - \frac{\mu}{r^3}(\boldsymbol{\rho} + \mathbf{d})$$

we have

$$\left(\frac{d^2\rho}{dt^2} + \frac{\mu}{r^3}\rho\right)\mathbf{i}_\rho + 2\frac{d\rho}{dt}\frac{d\mathbf{i}_\rho}{dt} + \rho\frac{d^2\mathbf{i}_\rho}{dt^2} + \mu\left(\frac{1}{r^3} - \frac{1}{d^3}\right)\mathbf{d} = \mathbf{0}$$

Next, take the scalar product of this vector equation with the vector cross-product

$$\mathbf{i}_\rho \times \frac{d\mathbf{i}_\rho}{dt}$$

to obtain

$$\rho\underbrace{\left(\mathbf{i}_\rho \times \frac{d\,\mathbf{i}_\rho}{dt}\cdot\frac{d^2\mathbf{i}_\rho}{dt^2}\right)}_{=D_1} + \mu\left(\frac{1}{r^3} - \frac{1}{d^3}\right)\underbrace{\left(\mathbf{i}_\rho \times \frac{d\,\mathbf{i}_\rho}{dt}\cdot\mathbf{d}\right)}_{=D_2} = 0$$

Similarly, using

$$\mathbf{i}_\rho \times \frac{d^2\mathbf{i}_\rho}{dt^2}$$

gives

$$2\frac{d\rho}{dt}\underbrace{\left(\mathbf{i}_\rho \times \frac{d^2\mathbf{i}_\rho}{dt^2}\cdot\frac{d\mathbf{i}_\rho}{dt}\right)}_{=-D_1} + \mu\left(\frac{1}{r^3} - \frac{1}{d^3}\right)\underbrace{\left(\mathbf{i}_\rho \times \frac{d^2\mathbf{i}_\rho}{dt^2}\cdot\mathbf{d}\right)}_{=D_3} = 0$$

As a result, we have **Laplace's equations**

$$\rho D_1 = \mu\left(\frac{1}{d^3} - \frac{1}{r^3}\right)D_2 \qquad (\mathbf{I})$$
$$2\frac{d\rho}{dt}D_1 = \mu\left(\frac{1}{r^3} - \frac{1}{d^3}\right)D_3 \qquad (\mathbf{II})$$
$$r^2 = \rho^2 + d^2 + 2\rho(\mathbf{i}_\rho\cdot\mathbf{d}) \qquad (\mathbf{III})$$

1. Solve (**I**) and (**III**) for r_2 and ρ_2.
2. Use (**II**) to determine $\dfrac{d\rho}{dt}$ at time t_2.

Lagrange's Interpolation Formulas (1778)

The vector $\mathbf{i}_\rho(t)$ can be expressed as a Taylor series

$$\mathbf{i}_\rho(t) = \mathbf{a}_0 + (t - t_0)\mathbf{a}_1 + \tfrac{1}{2}(t - t_0)^2\mathbf{a}_2 + O(t - t_0)^3$$

Expanding about the time t_2 gives

$$\begin{aligned}
\mathbf{i}_{\rho 3} &= \mathbf{a}_0 + \tau_1\mathbf{a}_1 + \tfrac{1}{2}\tau_1^2\mathbf{a}_2 \\
\mathbf{i}_{\rho 2} &= \mathbf{a}_0 \\
\mathbf{i}_{\rho 1} &= \mathbf{a}_0 - \tau_3\mathbf{a}_1 + \tfrac{1}{2}\tau_3^2\mathbf{a}_2
\end{aligned}$$

$$\mathbf{a}_0 = \mathbf{i}_\rho\Big|_{t=t_0} \qquad \mathbf{a}_1 = \frac{d\mathbf{i}_\rho}{dt}\Big|_{t=t_0} \qquad \mathbf{a}_2 = \frac{d^2\mathbf{i}_\rho}{dt^2}\Big|_{t=t_0}$$

Hence, we have

$$\begin{aligned}
\tau_1\mathbf{a}_1 + \tfrac{1}{2}\tau_1^2\mathbf{a}_2 &= \mathbf{i}_{\rho 3} - \mathbf{i}_{\rho 2} \\
\tau_3\mathbf{a}_1 - \tfrac{1}{2}\tau_3^2\mathbf{a}_2 &= \mathbf{i}_{\rho 2} - \mathbf{i}_{\rho 1}
\end{aligned}$$

to be solved for $\mathbf{a}_1$ and $\mathbf{a}_2$.

The result is

$$\begin{aligned}
\frac{d\mathbf{i}_\rho}{dt}\Big|_{t_2} &= -\frac{\tau_1}{\tau_2\tau_3}\,\mathbf{i}_{\rho 1} + \frac{\tau_1 - \tau_3}{\tau_1\tau_3}\,\mathbf{i}_{\rho 2} + \frac{\tau_3}{\tau_1\tau_2}\,\mathbf{i}_{\rho 3} \\
\frac{d^2\mathbf{i}_\rho}{dt^2}\Big|_{t_2} &= \frac{2}{\tau_2\tau_3}\,\mathbf{i}_{\rho 1} - \frac{2}{\tau_1\tau_3}\,\mathbf{i}_{\rho 2} + \frac{2}{\tau_1\tau_2}\,\mathbf{i}_{\rho 3}
\end{aligned}$$

which are valid to second order in the time intervals where

$$\tau_1 = t_3 - t_2 \qquad \tau_2 = t_3 - t_1 = \tau_1 + \tau_3 \qquad \tau_3 = t_2 - t_1$$

More accurate values for these derivatives can be obtained if more than three sets of observational data are available.

The determination of the derivatives of the observational data is the greatest weakness in Laplace's method of orbit determination. In fact, it is necessary to use additional observations to obtain any reasonable accuracy.