16.346 Astrodynamics
Fall 2008

Derivation of the Time Equation

Start with the Lagrange time equation and the equation for the mean point radius

$$\tfrac{1}{2}\sqrt{\mu}(t_2-t_1) = a^{\frac{3}{2}}(\psi - \sin\psi\cos\phi) \tag{1}$$
$$r_0 = a(1-\cos\phi) = r_{0p}(1+\tan^2\tfrac{1}{2}\psi) \tag{2}$$
$$r_{0p} = \tfrac{1}{2}\left[\tfrac{1}{2}(r_1+r_2) + \sqrt{r_1r_2}\cos\tfrac{1}{2}\theta\right] \tag{3}$$
$$FS = \sqrt{r_1r_2}\cos\tfrac{1}{2}\theta \tag{4}$$

Eliminate $\cos\phi$ and compare with the elementary form of Kepler's equation

$$\frac{1}{2}\sqrt{\frac{\mu}{a^3}}(t_2-t_1) = \psi - \sin\psi + \frac{r_0}{a}\sin\psi$$
$$\sqrt{\frac{\mu}{a^3}}(t-\tau) = E - \sin E + \frac{q}{a}\sin E$$

Identify

$$E \Longleftrightarrow \psi \qquad q \Longleftrightarrow r_0 \qquad t-\tau \Longleftrightarrow \tfrac{1}{2}(t_2-t_1)$$

since $t_2 - t_1 = (t_2-\tau)+(\tau-t_1) = 2(t_2-\tau)$.

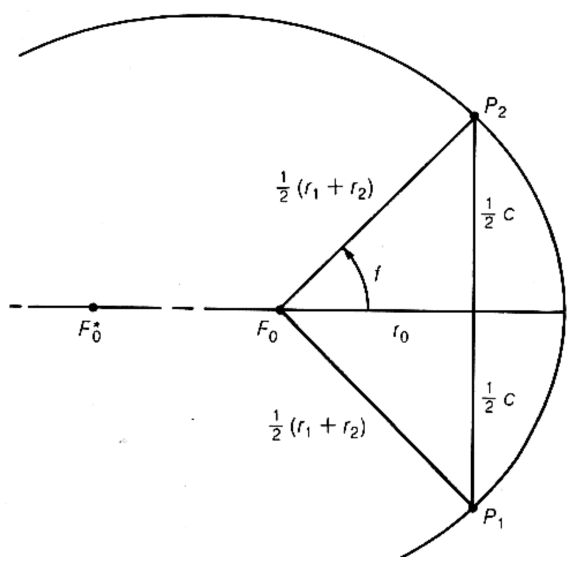

Fig. 7.4 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

From the classical relation between the true and eccentric anomalies:

$$\tan^2\tfrac{1}{2}f = \frac{1+e}{1-e}\tan^2\tfrac{1}{2}E = \frac{2a-q}{q}\tan^2\tfrac{1}{2}E \quad\Longrightarrow\quad \frac{q}{a} = \frac{2\tan^2\frac{1}{2}E}{\tan^2\frac{1}{2}f + \tan^2\frac{1}{2}E} = \frac{2x}{\ell + x}$$

where we have defined $\boxed{x = \tan^2\tfrac{1}{2}E}$ and $\boxed{\ell = \tan^2\tfrac{1}{2}f}$. Since

$$\cos f = \frac{F_0S}{F_0P_2} = \frac{\sqrt{r_1r_2}\cos\frac{1}{2}\theta}{\frac{1}{2}(r_1+r_2)} \quad\text{then}\quad \ell = \frac{1-\cos f}{1+\cos f} = \frac{F_0P_2 - F_0S}{F_0P_2 + F_0S}$$

Relation to Gauss' Classical Method

The time equation

$$\frac{1}{2}\sqrt{\frac{\mu}{q^3}}(t_2-t_1)\times\sqrt{\frac{q^3}{a^3}} = E-\sin E+\frac{q}{a}\sin E = E-\sin E+\frac{q}{a}\times\underbrace{\frac{2\tan\frac{1}{2}E}{1+\tan^2\frac{1}{2}E}}_{=\sin E}$$

becomes

$$\frac{1}{2}\sqrt{\frac{\mu}{q^3}}(t_2-t_1)\times\sqrt{\frac{8}{(\ell+x)^3}\times\tan^3\tfrac{1}{2}E} = E-\sin E+\frac{4\tan^3\frac{1}{2}E}{(\ell+x)(1+x)}$$

Then, since

$$q = r_0 = r_{0p}(1+\tan^2\tfrac{1}{2}E) = r_{0p}(1+x)$$

we have an expression for the transfer time as a function only of $E \equiv \psi$:

$$\sqrt{\frac{\mu}{8r_{0p}^3}}(t_2-t_1)\times\frac{4\tan^3\frac{1}{2}E}{[(\ell+x)(1+x)]^{\frac{3}{2}}} = E-\sin E+\frac{4\tan^3\frac{1}{2}E}{(\ell+x)(1+x)}$$

$$\sqrt{\frac{m^3}{[(\ell+x)(1+x)]^3}} = m\frac{E-\sin E}{4\tan^3\frac{1}{2}E}+\frac{m}{(\ell+x)(1+x)}$$

Following Gauss, we can define y as

$$\boxed{y^2 = \frac{m}{(\ell+x)(1+x)}} \qquad \text{so that} \qquad \boxed{y^3-y^2 = m\frac{E-\sin E}{4\tan^3\frac{1}{2}E}}$$

where

$$\boxed{m = \frac{\mu(t_2-t_1)^2}{8r_{0p}^3}} \qquad \text{and} \qquad \boxed{\ell = \frac{r_1+r_2-2\sqrt{r_1r_2}\cos\frac{1}{2}\theta}{r_1+r_2+2\sqrt{r_1r_2}\cos\frac{1}{2}\theta}}$$

Note: The new y does not have the same geometric significance as Gauss' y.

Parameter and Semimajor Axis

$$\frac{q}{a} = \frac{2x}{\ell+x} = \frac{r_{0p}(1+x)}{a} \qquad \Longrightarrow \qquad \frac{1}{a} = \frac{2x}{r_{0p}(1+x)(\ell+x)} = \frac{2xy^2}{mr_{0p}}$$

$$\frac{p}{p_m} = \frac{\sin\phi}{\sin\psi} = \frac{c}{2a\sin^2\psi} = \frac{c(1+x)^2}{8ax} = \frac{cy^2(1+x)^2}{4mr_{0p}} = \frac{y^2\sqrt{\ell}\,(1+x)^2}{m}$$

where we have used the equation

$$c = 2a\sin\psi\sin\phi$$

from **Lecture 9, Page 1**

Universal Form

The equations are universal when x is extended to include the other conics:

$$x = \begin{cases} \tan^2 \frac{1}{4}(E_2 - E_1) & \text{ellipse} \\ 0 & \text{parabola} \\ -\tanh^2 \frac{1}{4}(H_2 - H_1) & \text{hyperbola} \end{cases}$$

Comparing the Structure of Gauss' Method and the New Method

Gauss' Method

$$D \equiv \sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta$$
$$\ell = \frac{r_1 + r_2 - 2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta}{4D}$$
$$m = \frac{\mu(t_2 - t_1)^2}{8D^3}$$
$$x \equiv \sin^2\tfrac{1}{2}\psi$$
$$y^2 = \frac{m}{\ell + x}$$
$$y^3 - y^2 = m\,\frac{2\psi - \sin 2\psi}{\sin^3\psi}$$
$$\frac{p}{p_m} = \frac{cy^2}{4mD}$$
$$\frac{1}{a} = \frac{2y^2 x}{mD}(1 - x)$$

New Method

$$D \equiv \tfrac{1}{4}(r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta)$$
$$\ell = \frac{r_1 + r_2 - 2\sqrt{r_1 r_2}\cos\frac{1}{2}\theta}{4D}$$
$$m = \frac{\mu(t_2 - t_1)^2}{8D^3}$$
$$x \equiv \tan^2\tfrac{1}{2}\psi$$
$$y^2 = \frac{m}{(\ell + x)(1 + x)}$$
$$y^3 - y^2 = m\,\frac{\psi - \sin\psi}{4\tan^3\frac{1}{2}\psi}$$
$$\frac{p}{p_m} = \frac{cy^2}{4mD}(1 + x)^2$$
$$\frac{1}{a} = \frac{2y^2 x}{mD}$$