16.346 Astrodynamics
Fall 2008

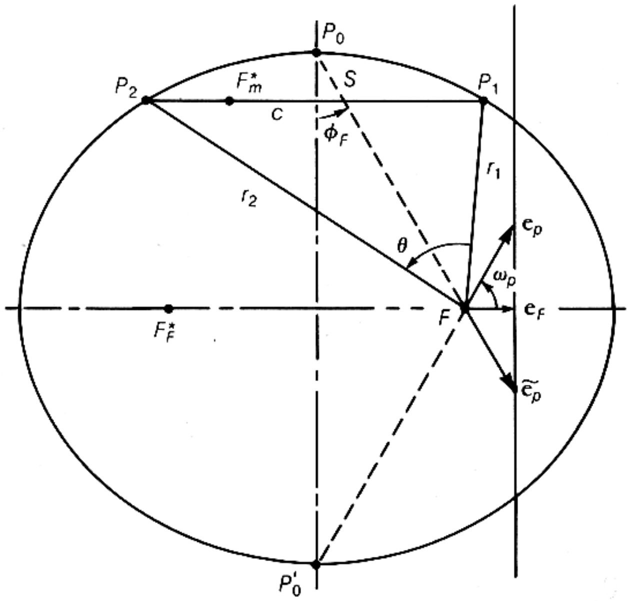


Fig. 6.13 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Recall: Equation of orbit $\mathbf{e}\cdot\mathbf{r} = p - r$ at P_1 and P_2 so that

$$\begin{aligned}\mathbf{e}\cdot\mathbf{r}_1 &= p - r_1\\ \mathbf{e}\cdot\mathbf{r}_2 &= p - r_2\end{aligned} \quad\Longrightarrow\quad \mathbf{e}\cdot(\mathbf{r}_2-\mathbf{r}_1) = r_1 - r_2 \qquad \text{or} \qquad \boxed{-\mathbf{e}\cdot\mathbf{i}_c = \frac{r_2-r_1}{c}}$$

Orbital Elements of the Fundamental (Minimum Eccentricity) Ellipse

$$\boxed{e_F = \frac{|r_2-r_1|}{c} \qquad a_F = \tfrac{1}{2}(r_1+r_2) \qquad \frac{p_F}{p_m} = \frac{r_1+r_2}{c}}$$

Page 258

Since $p_F = a_F(1-e_F^2)$ and

$$\begin{aligned}1 - e_F^2 &= \frac{c^2-(r_2-r_1)^2}{c^2} = \frac{1}{c^2}(c+r_2-r_1)(c+r_1-r_2)\\ &= \frac{4}{c^2}(s-r_1)(s-r_2) = \frac{4r_1r_2}{c^2}\sin^2\tfrac{1}{2}\theta = \frac{2p_m}{c}\end{aligned}$$

From the figure at the top of this page, we see that $\sin\phi_F = e_F$. Also the angle ω_p is the complement of ϕ_F so that $\cos\omega_p = e_F$. Therefore:

The axes of the conjugate parabolic orbits coincide with the lines through the focus F and the extremities of the minor axis of the fundamental ellipse.

Locus of Mean Points #6.4

Definition of **mean point:** At the point $\mathbf{r}_0$, the velocity $\mathbf{v}_0$ is parallel to the chord. Using the eccentricity vector at this point, we have $\mathbf{v}_0 \times \mathbf{h} \cdot (\mathbf{r}_2 - \mathbf{r}_1) = 0$ Therefore:

$$\mu \mathbf{e} \cdot (\mathbf{r}_2 - \mathbf{r}_1) = \left(\mathbf{v}_0 \times \mathbf{h} - \frac{\mu}{r_0} \mathbf{r}_0 \right) \cdot (\mathbf{r}_2 - \mathbf{r}_1) = 0 - \frac{\mu}{r_0} \mathbf{r}_0 \cdot (\mathbf{r}_2 - \mathbf{r}_1)$$

Hence: $$\underbrace{\mathbf{e} \cdot (\mathbf{r}_2 - \mathbf{r}_1)}_{= r_1 - r_2} = -\frac{1}{r_0} \mathbf{r}_0 \cdot (\mathbf{r}_2 - \mathbf{r}_1) \quad \Longrightarrow \quad \boxed{\mathbf{r}_0 \cdot (\mathbf{r}_2 - \mathbf{r}_1) = r_0 (r_2 - r_1)}$$

The loci of all mean points are the lines through the focus F and the extremities of the minor axis of the fundamental ellipse.

The Line Segment FS: Page 266

The line FS is the distance along mean point locus from the focus F to intersection with chord. The flight direction angle at P_0 is γ_0 and δ is the angle opposite the line segment SP_1. Use the law of sines for the triangles:

$$\triangle FP_1S: \quad \frac{FS}{\sin(\gamma_0 + \delta)} = \frac{r_1}{\sin \gamma_0} \qquad \triangle FP_1P_2: \quad \frac{c}{\sin \theta} = \frac{r_2}{\sin(\gamma_0 + \delta)}$$

and use the calculation on the previous page for $1 - e_F^2$:

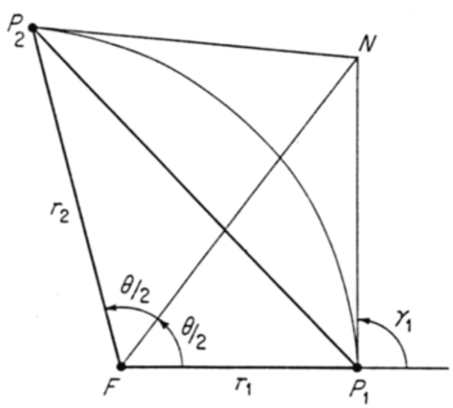
$$\triangle FP_0C: \quad \sin \gamma_0 = \frac{b_F}{a_F} = \sqrt{1 - e_F^2} = \frac{2}{c} \sqrt{r_1 r_2} \sin \tfrac{1}{2} \theta$$

Therefore:

$$\boxed{FS = \sqrt{r_1 r_2} \cos \tfrac{1}{2} \theta}$$

Recall the proposition: Lecture 8, Page 2

The line connecting the focus and the point of intersection of the orbital tangents at the terminals bisects the transfer angle.



$$\sqrt{r_1 r_2} = \begin{cases} FN \cos \frac{1}{2}(E_2 - E_1) & \text{ellipse} \\ FN & \text{parabola} \\ FN \cosh \frac{1}{2}(H_2 - H_1) & \text{hyperbola} \end{cases}$$

Fig. 6.7 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

For the ellipse $FN_1 \cos \frac{1}{2}(E_0 - E_1) = \sqrt{r_1 r_0}$ $FN_2 \cos \frac{1}{2}(E_2 - E_0) = \sqrt{r_0 r_2}$

and for the parabola $FN_{1p} = \sqrt{r_1 r_{0p}}$ $FN_{2p} = \sqrt{r_{0p} r_2}$

Triangle $\Delta F N_{1p} N_{2p}$ is similar to triangle $\Delta F N_1 N_2$. Therefore

$$\frac{FN_{1p}}{FN_{2p}} = \frac{FN_1}{FN_2} \quad \Longrightarrow \quad \cos \tfrac{1}{2}(E_0 - E_1) = \cos \tfrac{1}{2}(E_2 - E_0)$$

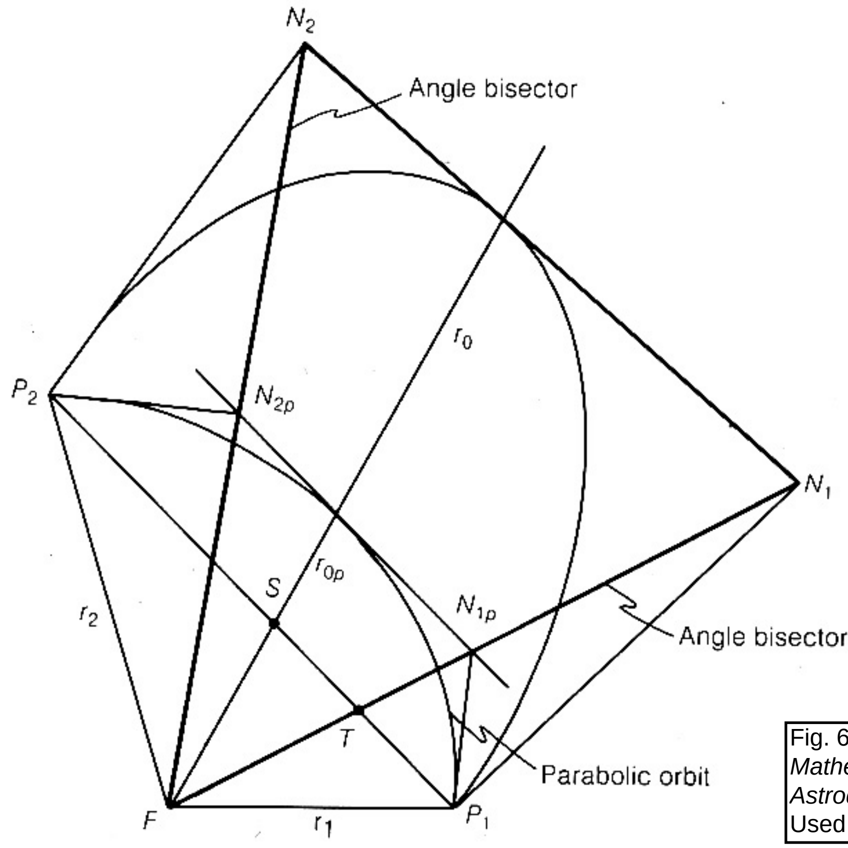


Fig. 6.16 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Hence

$$\boxed{E_0 = \tfrac{1}{2}(E_1 + E_2)} \qquad \textbf{6.73}$$

The eccentric anomaly of the mean point of an orbit connecting two termini is the arithmetic mean between the eccentric anomalies of those termini.

Mean-Point Radius of the Parabolic Orbit — Lecture 8, Page 1

The parameter of the parabola is obtained from

$$\left(\frac{p_p}{p_m}\right)^2 - 2D\frac{p_p}{p_m} + 1 = 0 \quad \text{where} \quad D = \frac{r_1 + r_2}{c} - \frac{s(s-c)}{ac} = \frac{r_1 + r_2}{c}$$

so that

$$\frac{p_p}{p_m} = \frac{1}{c}(r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta) = \frac{1}{c}\left(\sqrt{s} + \sqrt{s-c}\right)^2$$

The mean point radius of the parabola is

$$r_{0p} = \frac{p_p}{1 + \cos 2\phi_F} = \frac{p_p}{2\cos^2\phi_F} = \frac{p_p}{2(1 - e_F^2)} = \frac{p_p c}{4p_m}$$

Hence

$$\boxed{r_{0p} = \tfrac{1}{4}(r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta) = \tfrac{1}{2}(a_F + FS)}$$

The mean point radius of the parabola extends to the midpoint between the chord and the extremity of the minor axis of the fundamental ellipse.

Mean-Point Radius of Ellipic and Hyperbolic Orbits — Page 270

From the derivation of the eccentric anomaly of the mean point, we have

$$FN_2\cos\tfrac{1}{2}(E_2 - E_0) = FN_2\cos\tfrac{1}{2}[E_2 - \tfrac{1}{2}(E_1 + E_2)] = FN_2\cos\tfrac{1}{4}(E_2 - E_1) = \sqrt{r_0 r_2}$$

$$FN_{2p} = \sqrt{r_{0p} r_2}$$

so that

$$\frac{FN_2}{FN_{2p}}\cos\tfrac{1}{4}(E_2 - E_1) = \sqrt{\frac{r_0}{r_{0p}}}$$

But, from similar triangles,

$$\frac{FN_2}{FN_{2p}} = \frac{r_0}{r_{0p}}$$

Therefore, we have the truly elegant expression

$$r_0 = r_{0p}\sec^2\tfrac{1}{4}(E_2 - E_1) = r_{0p}\sec^2\tfrac{1}{2}\psi = r_{0p}(1 + \tan^2\tfrac{1}{2}\psi)$$

and, as we might expect,

$$r_0 = r_{0p}\,\mathrm{sech}^2\,\tfrac{1}{4}(H_2 - H_1)$$

obtains also for hyperbolic orbits.

$$\boxed{r_0 = \begin{cases} a[1 - e\cos\frac{1}{2}(E_1 + E_2)] = a(1 - \cos\phi) \\ a[1 - e\cosh\frac{1}{2}(H_1 + H_2)] \end{cases} = \begin{cases} r_{0p}(1 + \tan^2\frac{1}{2}\psi) \\ r_{0p}(1 - \tanh^2\frac{1}{4}(H_2 - H_1) \end{cases}}$$