16.346 Astrodynamics
Fall 2008

## The Theorem of Johann Heinrich Lambert #6.6

$$\boxed{\sqrt{\mu}\,(t_2 - t_1) = F(a, r_1 + r_2, c)}$$

## Developing Lagrange's Equations

- Kepler's Equation $\sqrt{\mu}\,(t - \tau) = a^{\frac{3}{2}}(E - e\sin E)$

$$\sqrt{\mu}\,(t_2 - \tau) - \sqrt{\mu}\,(t_1 - \tau) = a^{\frac{3}{2}}[(E_2 - e\sin E_2) - (E_1 - e\sin E_1)$$

$$\sqrt{\mu}\,(t_2 - t_1) = 2a^{\frac{3}{2}}[\tfrac{1}{2}(E_2 - E_1) - e\sin\tfrac{1}{2}(E_2 - E_1)\cos\tfrac{1}{2}(E_1 + E_2)]$$

- Equation of Orbit $r = a(1 - e\cos E)$

$$\begin{aligned} r_1 + r_2 &= a(1 - e\cos E_1) + a(1 - e\cos E_2) \\ &= 2a[1 - e\cos\tfrac{1}{2}(E_2 - E_1)\cos\tfrac{1}{2}(E_1 + E_2)] \end{aligned}$$

- Chord $c = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}$

$$\begin{aligned} c^2 &= r_1^2 + r_2^2 + 2r_1r_2(1 - 2\cos^2\tfrac{1}{2}\theta) \\ &= (r_1 + r_2)^2 - 4r_1r_2\cos^2\tfrac{1}{2}\theta \end{aligned}$$

Recall the relations between eccentric and true anomalies.

Using $\sqrt{r}\sin\frac{1}{2}f = \sqrt{a(1+e)}\sin\frac{1}{2}E \qquad \sqrt{r}\cos\frac{1}{2}f = \sqrt{a(1-e)}\cos\frac{1}{2}E$

$$\begin{aligned} \sqrt{r_1r_2}\cos\tfrac{1}{2}\theta &= \sqrt{r_1r_2}\cos\tfrac{1}{2}(f_2 - f_1) = \sqrt{r_1r_2}(\cos\tfrac{1}{2}f_2\cos\tfrac{1}{2}f_1 + \sin\tfrac{1}{2}f_2\sin\tfrac{1}{2}f_1) \\ &= \sqrt{r_2}\cos\tfrac{1}{2}f_2\,\sqrt{r_1}\cos\tfrac{1}{2}f_1 + \sqrt{r_2}\sin\tfrac{1}{2}f_2\,\sqrt{r_1}\sin\tfrac{1}{2}f_1 \\ &= a(1-e)\cos\tfrac{1}{2}E_2\cos\tfrac{1}{2}E_1 + a(1+e)\sin\tfrac{1}{2}E_2\sin\tfrac{1}{2}E_1 \\ &= a\cos\tfrac{1}{2}(E_2 - E_1) - ae\cos\tfrac{1}{2}(E_2 + E_1) \end{aligned}$$

- Lagrange Parameters

$$\boxed{\begin{aligned} \psi &= \tfrac{1}{2}(E_2 - E_1) \\ \cos\phi &= e\cos\tfrac{1}{2}(E_1 + E_2) \end{aligned}}$$

- Lagrange Equations

$$\boxed{\begin{aligned} \sqrt{\mu}(t_2 - t_1) &= 2a^{\frac{3}{2}}(\psi - \sin\psi\cos\phi) \\ r_1 + r_2 &= 2a(1 - \cos\psi\cos\phi) \\ c &= 2a\sin\psi\sin\phi \end{aligned}}$$

Note: We also have $\boxed{\sqrt{r_1r_2}\cos\frac{1}{2}\theta = a(\cos\psi - \cos\phi)}$

## Lagrange's Time Equation

Define

$$\begin{aligned} \alpha &= \phi + \psi \\ \beta &= \phi - \psi \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \psi &= \tfrac{1}{2}(\alpha - \beta) \\ \phi &= \tfrac{1}{2}(\alpha + \beta) \end{aligned}$$

Then

$$\begin{aligned} \sqrt{\mu}(t_2 - t_1) &= 2a^{\frac{3}{2}}(\psi - \sin\psi\cos\phi) \\ &= a^{\frac{3}{2}}[\alpha - \beta - 2\sin\tfrac{1}{2}(\alpha-\beta)\cos\tfrac{1}{2}(\alpha+\beta)] \\ &= a^{\frac{3}{2}}[(\alpha - \sin\alpha) - (\beta - \sin\beta)] \end{aligned}$$

Also:

$$\begin{aligned} r_1 + r_2 + c &= 2a[1 - \cos(\phi+\psi)] = 2a(1-\cos\alpha) = 4a\sin^2\tfrac{1}{2}\alpha \\ r_1 + r_2 - c &= 2a[1 - \cos(\phi-\psi)] = 2a(1-\cos\beta) = 4a\sin^2\tfrac{1}{2}\beta \end{aligned}$$

Hence, Lagrange's analytic form of Lambert's theorem is

$$\boxed{\sqrt{\frac{\mu}{a^3}}(t_2 - t_1) = (\alpha - \sin\alpha) - (\beta - \sin\beta)}$$

where

$$\boxed{\sin^2\tfrac{1}{2}\alpha = \frac{s}{2a} \qquad \sin^2\tfrac{1}{2}\beta = \frac{s-c}{2a}}$$

in terms of the semiperimeter of the triangle:

$$\boxed{s = \tfrac{1}{2}(r_1 + r_2 + c)}$$

## Euler's Equation for Parabolic Orbits

Since

$$a^{\frac{3}{2}}(\alpha - \sin\alpha) = a^{\frac{3}{2}}\left(\frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \cdots\right)$$

and

$$\alpha = 2\arcsin\sqrt{\frac{s}{2a}} = 2\left(\frac{s}{2a}\right)^{\frac{1}{2}} + \frac{1}{3}\left(\frac{s}{2a}\right)^{\frac{3}{2}} + \cdots$$

Then

$$a^{\frac{3}{2}}(\alpha - \sin\alpha) = \frac{\sqrt{2}}{3}s^{\frac{3}{2}} + O\left(\frac{1}{a^2}\right)$$

Similarly,

$$a^{\frac{3}{2}}(\beta - \sin\beta) = \frac{\sqrt{2}}{3}(s-c)^{\frac{3}{2}} + O\left(\frac{1}{a^2}\right)$$

Therefore:

$$\sqrt{\mu}(t_2 - t_1) = \frac{\sqrt{2}}{3}[s^{\frac{3}{2}} \mp (s-c)^{\frac{3}{2}}]$$

The choice of sign is minus for $\theta < 180°$ and plus for $\theta > 180°$.

Alternately,

$$\boxed{6\sqrt{\mu}(t_2 - t_1) = (r_1 + r_2 + c)^{\frac{3}{2}} \mp (r_1 + r_2 - c)^{\frac{3}{2}}}$$

## The Orbital Parameter

From Page 1 of Lecture 8

$$\left(\frac{p}{p_m}\right)^2 - 2D\frac{p}{p_m} + 1 = 0 \qquad \text{where} \qquad D = \frac{r_1 + r_2}{c} - \frac{r_1 r_2}{ac}\cos^2 \tfrac{1}{2}\theta$$

Use the Lagrange equations

$$D = \frac{2a(1 - \cos\psi\cos\phi)}{2a\sin\psi\sin\phi} - \frac{a^2(\cos\psi - \cos\phi)^2}{2a^2\sin\psi\sin\phi} = \frac{\sin^2\psi + \sin^2\phi}{2\sin\psi\sin\phi}$$

$$\sqrt{D^2 - 1} = \frac{\sin^2\psi - \sin^2\phi}{2\sin\psi\sin\phi}$$

so that

$$\frac{p}{p_m} = \begin{cases} \dfrac{\sin\phi}{\sin\psi} = \dfrac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)} \\[2ex] \dfrac{\sin\psi}{\sin\phi} = \dfrac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)} \end{cases}$$

Note: The orbital parameter equations were not developed by Lagrange.

## Skewed-Velocity Components in Terms of the Semimajor Axis #6.8

From

$$p = \frac{\sin\phi}{\sin\psi}\,p_m = \frac{\sin\phi}{\sin\psi} \times \frac{r_1 r_2}{c}(1 - \cos\theta)$$

and

$$c = 2a\sin\psi\sin\phi \qquad \alpha = \phi + \psi$$

$$\sqrt{r_1 r_2}\cos\tfrac{1}{2}\theta = a(\cos\psi - \cos\phi) \qquad \beta = \phi - \psi$$

we obtain

$$v_c = \frac{c\sqrt{\mu p}}{r_1 r_2 \sin\theta} = \sqrt{\frac{\mu}{a}}\,\frac{\sin\phi}{\cos\psi - \cos\phi} = \sqrt{\frac{\mu}{4a}}(\cot\tfrac{1}{2}\beta + \cot\tfrac{1}{2}\alpha)$$

$$v_\rho = \sqrt{\frac{\mu}{p}}\,\frac{1 - \cos\theta}{\sin\theta} = \sqrt{\frac{\mu}{a}}\,\frac{\sin\psi}{\cos\psi - \cos\phi} = \sqrt{\frac{\mu}{4a}}(\cot\tfrac{1}{2}\beta - \cot\tfrac{1}{2}\alpha)$$

Hence:

$$\mathbf{v}_1 = \left(\sqrt{\frac{\mu}{2(s-c)} - \frac{\mu}{4a}} + \sqrt{\frac{\mu}{2s} - \frac{\mu}{4a}}\right)\mathbf{i}_c + \left(\sqrt{\frac{\mu}{2(s-c)} - \frac{\mu}{4a}} - \sqrt{\frac{\mu}{2s} - \frac{\mu}{4a}}\right)\mathbf{i}_{r_1}$$