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16.346 Astrodynamics
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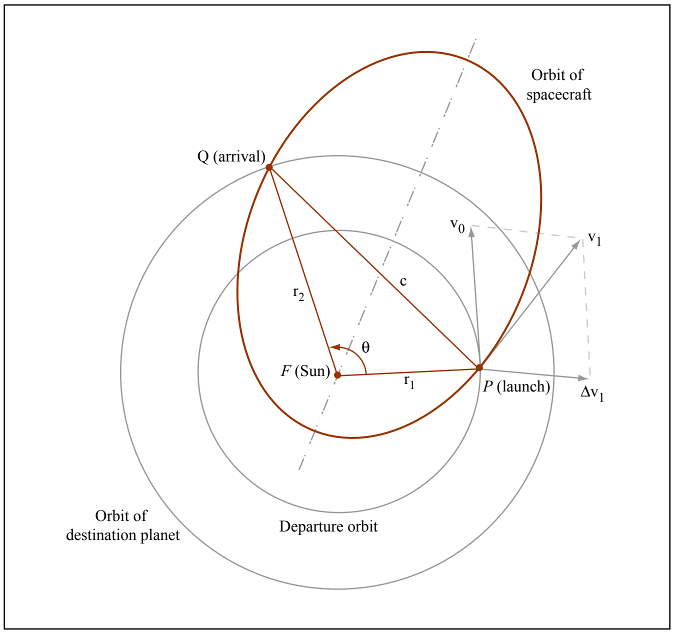
Velocity Requirements for Orbital Transfer



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Optimum Single-Impulse Transfer #11.2

We can apply Euler's tangent to the hyperbola, derived in Lecture 6, to formulate the solution of the optimum velocity impulse problem.

$$\mathbf{v}_0 = \text{initial velocity of spacecraft at } P$$
$$\Delta\mathbf{v}_1 = \mathbf{v}_1 - \mathbf{v}_0 = \mathbf{v}_{\rho_1} + \mathbf{v}_{c_1} - \mathbf{v}_0 = \text{required velocity impulse}$$

Then the necessary and sufficient condition for an optimum transfer is:

$$\boxed{\Delta\mathbf{v}_1 \perp (\mathbf{v}_{\rho_1} - \mathbf{v}_{c_1})} \quad \Longrightarrow \quad (\mathbf{v}_{\rho_1} - \mathbf{v}_{c_1}) \cdot (\mathbf{v}_{\rho_1} + \mathbf{v}_{c_1} - \mathbf{v}_0) = 0$$

To convert this to an algebraic equation, we have

$$\left(\frac{v_{\rho_1}}{v_{c_1}}\right)^2 - 1 - \frac{1}{v_{c_1}}\left[\frac{v_{\rho_1}}{v_{c_1}}(\mathbf{v}_0 \cdot \mathbf{i}_{r_1}) - (\mathbf{v}_0 \cdot \mathbf{i}_c)\right] = 0$$

Define

$$x^2 = \frac{v_{\rho_1}}{v_{c_1}} = \frac{p_m}{p} \qquad \text{where} \qquad p_m = \frac{2r_1r_2}{c}\sin^2\tfrac{1}{2}\theta$$

Then

$$\frac{1}{v_{c_1}} = \frac{r_1r_2\sin\theta}{c\sqrt{\mu p}} = \sqrt{\frac{2r_1r_2}{\mu c}}\,x\cos\tfrac{1}{2}\theta \qquad \text{where} \qquad x > 0$$

Optimality condition:

$$x^4 - \sqrt{\frac{2r_1r_2}{\mu c}} \cos \tfrac{1}{2}\theta\, (\mathbf{v}_0 \cdot \mathbf{i}_{r_1})\, x^3 + \sqrt{\frac{2r_1r_2}{\mu c}} \cos \tfrac{1}{2}\theta\, (\mathbf{v}_0 \cdot \mathbf{i}_c)\, x - 1 = 0$$

We have

$$\mathbf{v}_0 \cdot \mathbf{i}_{r_1} = v_0 \cos\gamma_0 = \frac{\sqrt{\mu p_0}}{r_1} \cot\gamma_0$$

$$\mathbf{v}_0 \cdot \mathbf{i}_c = v_0 \cos(\phi_1 - \gamma_0) = \frac{\sqrt{\mu p_0}}{r_1} (\cos\phi_1 \cot\gamma_0 + \sin\phi_1)$$

Define

$$P = \sqrt{\frac{2r_2p_0}{r_1c}} \cos \tfrac{1}{2}\theta \cot\gamma_0 \qquad Q = \sqrt{\frac{2r_2p_0}{r_1c}} \cos \tfrac{1}{2}\theta\, (\cos\phi_1 \cot\gamma_0 + \sin\phi_1)$$

Then $\boxed{x^4 - Px^3 + Qx - 1 = 0}$

Optimum Single-Impulse Transfer from a Circular Orbit

$$\boxed{x^4 + Qx - 1 = 0} \qquad \text{with} \qquad Q = \sqrt{\frac{2r_2^3}{c^3}} \cos \tfrac{1}{2}\theta \sin\theta$$

The result of Problem 11–4 in the textbook is that the optimum single impulse transfer from a circular orbit can be expressed as

$$\sin\nu - \tan\nu = \frac{4}{Q} = \frac{4(c/r_2)^{\frac{3}{2}}}{\sin\theta\sqrt{1+\cos\theta}}$$

where the angle ν, introduced in Section 6.5, is given by

$$x^2 = \frac{p_m}{p} = \cot^2 \tfrac{1}{2}\nu$$

and $0 \le \nu \le \pi$. In this form, the equation can be solved for ν almost by inspection using a table of trigonometric functions.

One of my past students, Peter Neirinckx, showed that this equation could be solved by successive substitutions. Peter's argument was that since ν is in the second quadrant, the transformation $\nu = \pi + \alpha$ would be convenient and the algorithm could be expressed recursively in the form:

$$\boxed{\alpha_{n+1} = \arctan[-(\sin\alpha_n + 4/Q)]}$$

His initial guess was $\alpha_0 = -45°$ and convergence was very rapid indeed.

The exact solution is developed on Pages 521–522.

Solving the General Optimization Problem by Successive Substitutions

A more recent student, Phil Springmann, in 2003, showed that the general case is similar in form to the optimum transfer from a circular orbit. Phil's development is

$$x^4 - Px^3 + Qx - 1 = 0$$

$$\cot^4 \tfrac{1}{2}\nu - P\cot^3 \tfrac{1}{2}\nu + Q\cot \tfrac{1}{2}\nu - 1 = 0$$

$$(\cos^4 \tfrac{1}{2}\nu - \sin^4 \tfrac{1}{2}\nu) - P\cos^3 \tfrac{1}{2}\nu \sin \tfrac{1}{2}\nu + Q\cos \tfrac{1}{2}\nu \sin^3 \tfrac{1}{2}\nu = 0$$

$$\cos\nu - \tfrac{1}{4}P[\sin\nu(1+\cos\nu)] + \tfrac{1}{4}Q[\sin\nu(1-\cos\nu)] = 0$$

$$(Q+P)\sin\nu - (Q-P)\tan\nu = 4$$

Then, as a recursive algorithm, with $\nu = \pi + \alpha$

$$\boxed{\alpha_{n+1} = \arctan\left\{\frac{-[(Q+P)\sin\alpha_n + 4]}{Q-P}\right\}}$$

Earth-to-Mars Departure Velocity for Orbital Transfer

Introduce the dimensionless quantity

$$\Delta E = \left(\frac{\Delta v_1}{v_0}\right)^2$$

which is the amount of energy needed at point P_1 to transfer from a circular orbit to an elliptical orbit for a voyage to P_2. We find that

1. ΔE is a double-valued function of a having an infinite slope at $a = a_m = \frac{1}{2}s$ where

$$s = \tfrac{1}{2}(r_1 + r_2 + c)$$

which is the smallest value of a for which an elliptical path from P_1 to P_2 is possible.

2. As a increases, the slope of the upper branch of ΔE is always positive, while the slope of the lower branch is negative for a near a_m and has a minimum for $a = a_M$. Both branches then approach horizontal asymptotes.

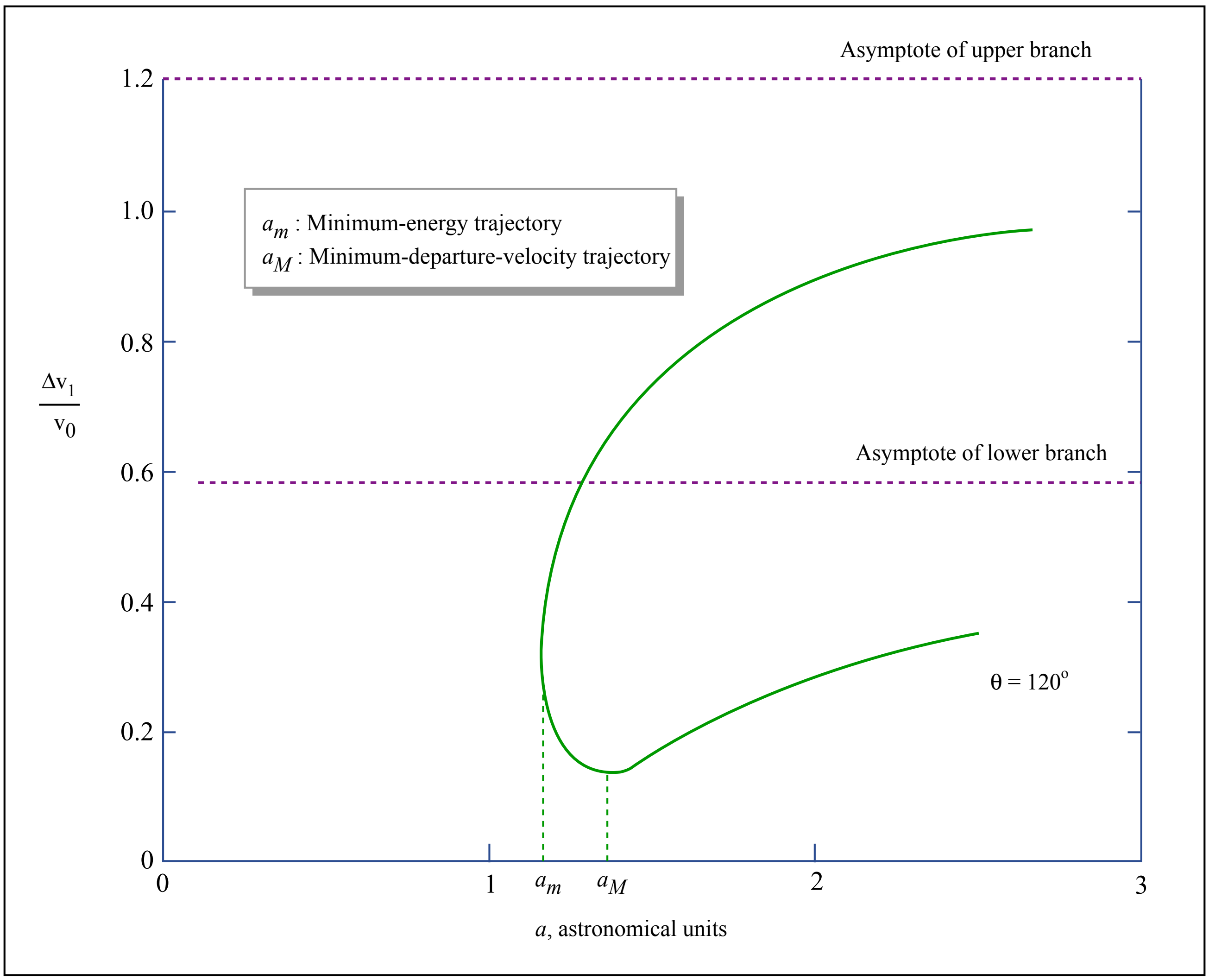


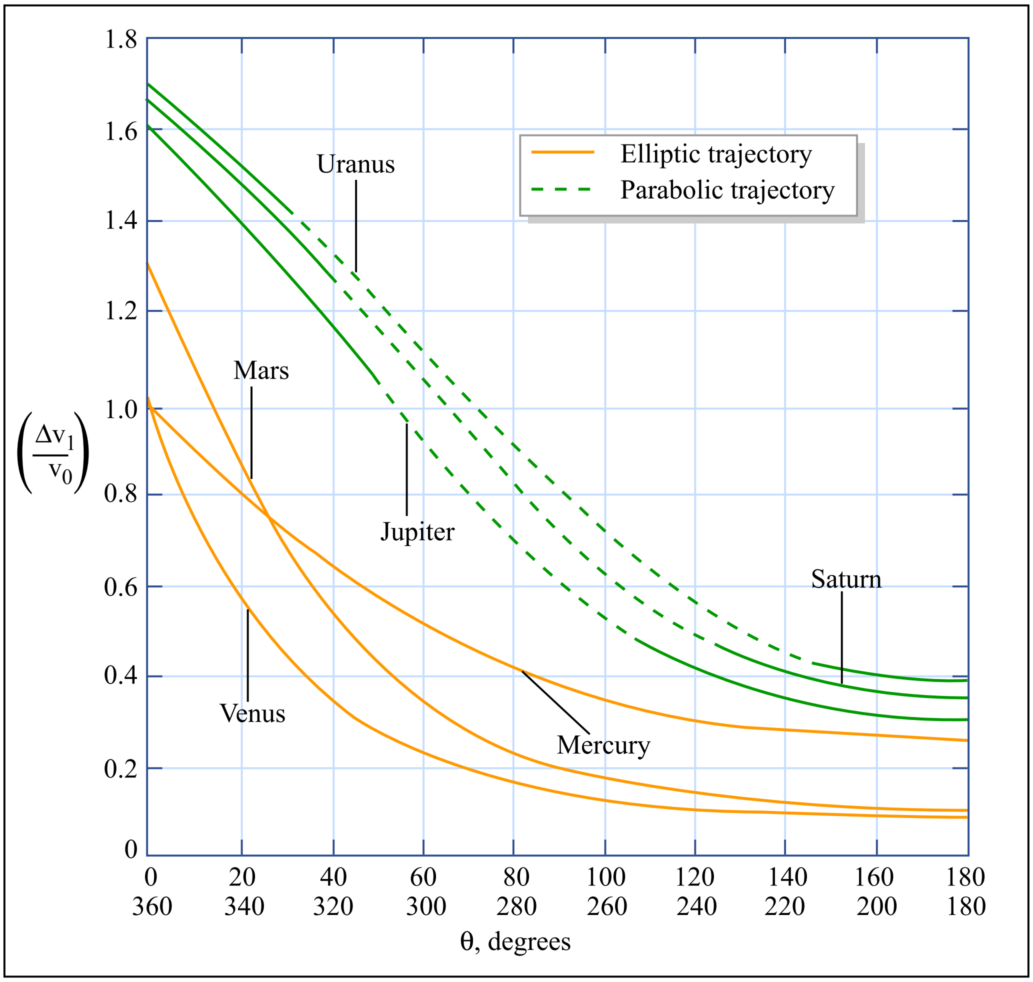
Figure by MIT OpenCourseWare.

Minimum Departure Velocity for Orbital Transfer from Earth to the Planets

The figure below gives plots of

$$\left(\frac{\Delta v_1}{v_0}\right)_M$$

as a function of the transfer angle θ for a voyage from earth to each of the other planets of the solar system. The curves for the two most remote planets Neptune and Pluto are not shown because, with the scale used, they would be indistinguishable from the curve for Uranus. The sections of the curves for Jupiter, Saturn and Uranus corresponding to parabolic trajectories as the minimum-velocity paths, are characterized by broken lines. For the special case in which $\theta = 180°$, the trajectory is of the Hohmann type. (For transfer angles θ larger than $180°$, the parabolic trajectories are ficitious optimums since they would not be closed in the counterclockwise direction.)



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