16.346 Astrodynamics
Fall 2008

Euler Angles Ω, i, ω

Define two sets of orbital plane unit vectors:

(1) $\mathbf{i}_e \equiv \mathbf{i}_\xi$ and $\mathbf{i}_p \equiv \mathbf{i}_\eta$ referenced to pericenter, and

(2) $\mathbf{i}_n$ and $\mathbf{i}_m$ referenced to the ascending node.

with $\mathbf{i}_h \equiv \mathbf{i}_\zeta$. Each of these can be expressed in terms of the Euler angles as components along the reference axes unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, $\mathbf{i}_z$ using vector algebra.

We have:

$$\begin{aligned}
\mathbf{i}_n &= \cos\Omega\,\mathbf{i}_x + \sin\Omega\,\mathbf{i}_y \\
\mathbf{i}_h &= \sin i\,\mathbf{i}_n \times \mathbf{i}_z + \cos i\,\mathbf{i}_z = \sin\Omega\sin i\,\mathbf{i}_x - \cos\Omega\sin i\,\mathbf{i}_y + \cos i\,\mathbf{i}_z \\
&= \ell_3\,\mathbf{i}_x + m_3\,\mathbf{i}_y + n_3\,\mathbf{i}_z \\
\mathbf{i}_m &= \mathbf{i}_h \times \mathbf{i}_n = -\sin\Omega\cos i\,\mathbf{i}_x + \cos\Omega\cos i\,\mathbf{i}_y + \sin i\,\mathbf{i}_z
\end{aligned}$$

$$\begin{aligned}
\mathbf{i}_e &= \cos\omega\,\mathbf{i}_n + \sin\omega\,\mathbf{i}_m \\
&= (\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i)\,\mathbf{i}_x + (\sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i)\,\mathbf{i}_y + \sin\omega\sin i\,\mathbf{i}_z \\
&= \ell_1\,\mathbf{i}_x + m_1\,\mathbf{i}_y + n_1\,\mathbf{i}_z \\
\mathbf{i}_p &= \mathbf{i}_h \times \mathbf{i}_e \\
&= -(\cos\Omega\sin\omega + \sin\Omega\cos\omega\cos i)\,\mathbf{i}_x - (\sin\Omega\sin\omega - \cos\Omega\cos\omega\cos i)\,\mathbf{i}_y + \cos\omega\sin i\,\mathbf{i}_z \\
&= \ell_2\,\mathbf{i}_x + m_2\,\mathbf{i}_y + n_2\,\mathbf{i}_z
\end{aligned}$$

so that

$$\begin{bmatrix} \mathbf{i}_e \\ \mathbf{i}_p \\ \mathbf{i}_h \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{bmatrix}$$

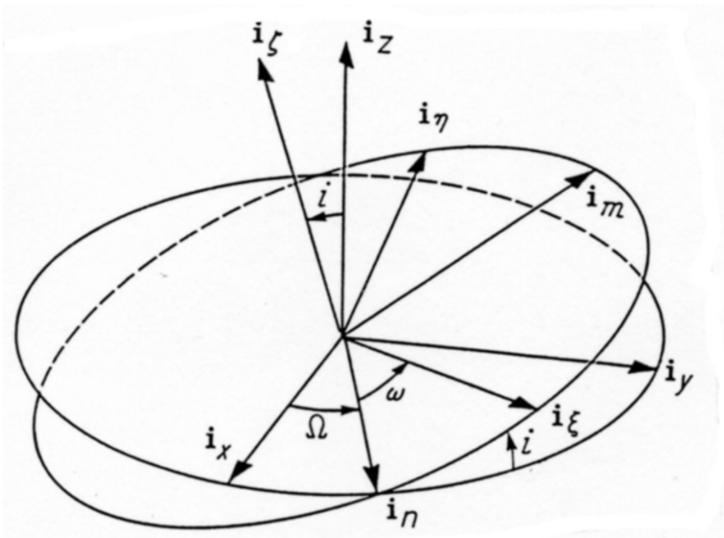

Fig. 2.2 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Position and Velocity Vectors in Reference Coordinates

See Problem 3–21

The rotation matrix to transform between orbital plane and reference coordinates is

$$\mathbf{R} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix}$$

with

$$\mathbf{r}_{orbital\,plane} = r\begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix} \qquad \mathbf{v}_{orbital\,plane} = \frac{\mu}{h}\begin{bmatrix} -\sin f \\ e + \cos f \\ 0 \end{bmatrix}$$

The proper expressions for position and velocity in reference coordinates are obtained by premultiplying the position and velocity vectors, expressed in orbital plane coordinates, by the rotation matrix $\mathbf{R}$. We also need the trigonometric identities

$$\sin\theta = \sin(\omega + f) = \sin\omega\cos f + \cos\omega\sin f$$
$$\cos\theta = \cos(\omega + f) = \cos\omega\cos f - \sin\omega\sin f$$

to obtain the position and velocity vectors in the form:

$$\begin{aligned} \mathbf{r} = {} & r(\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i)\,\mathbf{i}_x \\ & + r(\sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i)\,\mathbf{i}_y \\ & + r\sin\theta\sin i\,\mathbf{i}_z \end{aligned}$$

and

$$\begin{aligned} \mathbf{v} = {} & -\frac{\mu}{h}[\cos\Omega(\sin\theta + e\sin\omega) + \sin\Omega(\cos\theta + e\cos\omega)\cos i]\,\mathbf{i}_x \\ & -\frac{\mu}{h}[\sin\Omega(\sin\theta + e\sin\omega) - \cos\Omega(\cos\theta + e\cos\omega)\cos i]\,\mathbf{i}_y \\ & +\frac{\mu}{h}(\cos\theta + e\cos\omega)\sin i\,\mathbf{i}_z \end{aligned}$$

where $\theta = \omega + f$ is called the argument of latitude.

Terminology

#3.4, Page 160–161

Time of pericenter passage	τ	Argument of latitude	$\theta = \omega + f$
Angle of inclination	i	True longitude	$L = \varpi + f$
Longitude of ascending node	Ω	Mean anomaly	$M = n(t - \tau)$
Argument of pericenter	ω	Mean longitude	$l = \varpi + M = nt + \epsilon$
Longitude of pericenter	$\varpi = \Omega + \omega$	Mean longitude at epoch	$\epsilon = \varpi - n\tau$

Then the mean anomaly is determined from:

$$\boxed{M = nt + \epsilon - \varpi} \qquad (4.39)$$