16.333: Lecture # 7

Approximate Longitudinal Dynamics Models

- A couple more stability derivatives
- Given mode shapes found identify simpler models that capture the main responses

More Stability Derivatives

- Recall from 6–2 that the derivative stability derivative terms $Z_{\dot{w}}$ and $M_{\dot{w}}$ ended up on the LHS as modifications to the normal mass and inertia terms
  - These are the *apparent mass* effects – some of the surrounding displaced air is "entrained" and moves with the aircraft
  - Acceleration derivatives quantify this effect
  - Significant for blimps, less so for aircraft.

- Main effect: rate of change of the normal velocity $\dot{w}$ causes a transient in the downwash ϵ from the wing that creates a change in the angle of attack of the tail some time later – **Downwash Lag** effect

- If aircraft flying at U_0, will take approximately $\Delta t = l_t/U_0$ to reach the tail.
  - Instantaneous downwash at the tail $\epsilon(t)$ is due to the wing α at time $t - \Delta t$.

$$\epsilon(t) = \frac{\partial \epsilon}{\partial \alpha}\alpha(t - \Delta t)$$

  - Taylor series expansion

$$\alpha(t - \Delta t) \approx \alpha(t) - \dot{\alpha}\Delta t$$

  - Note that $\Delta\epsilon(t) = -\Delta\alpha_t$. Change in the tail AOA can be computed as

$$\Delta\epsilon(t) = -\frac{d\epsilon}{d\alpha}\dot{\alpha}\Delta t = -\frac{d\epsilon}{d\alpha}\dot{\alpha}\frac{l_t}{U_0} = -\Delta\alpha_t$$

- For the tail, we have that the lift increment due to the change in downwash is

$$\Delta C_{L_t} = C_{L_{\alpha_t}} \Delta \alpha_t = C_{L_{\alpha_t}} \dot{\alpha} \frac{d\epsilon}{d\alpha} \frac{l_t}{U_0}$$

The change in lift force is then

$$\Delta L_t = \frac{1}{2} \rho (U_0^2)_t S_t \Delta C_{L_t}$$

- In terms of the Z-force coefficient

$$\Delta C_Z = -\frac{\Delta L_t}{\frac{1}{2}\rho U_0^2 S} = -\eta \frac{S_t}{S} \Delta C_{L_t} = -\eta \frac{S_t}{S} C_{L_{\alpha_t}} \dot{\alpha} \frac{d\epsilon}{d\alpha} \frac{l_t}{U_0}$$

- We use $\bar{c}/(2U_0)$ to nondimensionalize time, so the appropriate stability coefficient form is (note use C_z to be general, but we are looking at ΔC_z from before):

$$\begin{aligned} C_{Z_{\dot{\alpha}}} = \left(\frac{\partial C_Z}{\partial \left(\dot{\alpha} \bar{c} / 2U_0 \right)} \right)_0 &= \frac{2U_0}{\bar{c}} \left(\frac{\partial C_Z}{\partial \dot{\alpha}} \right)_0 \\ &= -\eta \frac{2U_0}{\bar{c}} \frac{S_t}{S} \frac{l_t}{U_0} C_{L_{\alpha_t}} \frac{d\epsilon}{d\alpha} \\ &= -2\eta V_H C_{L_{\alpha_t}} \frac{d\epsilon}{d\alpha} \end{aligned}$$

- The pitching moment due to the lift increment is

$$\begin{aligned} \Delta M_{cg} &= -l_t \Delta L_t \\ \rightarrow \Delta C_{M_{cg}} &= -l_t \frac{\frac{1}{2}\rho (U_0^2)_t S_t \Delta C_{L_t}}{\frac{1}{2}\rho U_0^2 S \bar{c}} \\ &= -\eta V_H \Delta C_{L_t} = -\eta V_H C_{L_{\alpha_t}} \dot{\alpha} \frac{d\epsilon}{d\alpha} \frac{l_t}{U_0} \end{aligned}$$

- So that

$$
\begin{aligned}
C_{M_{\dot\alpha}} = \left(\frac{\partial C_M}{\partial\left(\dot\alpha\bar c/2U_0\right)}\right)_0 &= \frac{2U_0}{\bar c}\left(\frac{\partial C_M}{\partial\dot\alpha}\right)_0 \\
&= -\eta V_H C_{L_{\alpha_t}}\frac{d\epsilon}{d\alpha}\frac{l_t}{U_0}\frac{2U_0}{\bar c} \\
&= -2\eta V_H C_{L_{\alpha_t}}\frac{d\epsilon}{d\alpha}\frac{l_t}{\bar c} \\
&\equiv \frac{l_t}{\bar c}C_{Z_{\dot\alpha}}
\end{aligned}
$$

- Similarly, pitching motion of the aircraft changes the AOA of the tail. Nose pitch up at rate q, increases apparent downwards velocity of tail by ql_t, changing the AOA by

$$
\Delta\alpha_t = \frac{ql_t}{U_0}
$$

which changes the lift at the tail (and the moment about the cg).

- Following same analysis as above: Lift increment

$$
\begin{aligned}
\Delta L_t &= C_{L_{\alpha_t}}\frac{ql_t}{U_0}\frac{1}{2}\rho(U_0^2)_t S_t \\
\Delta C_Z &= -\frac{\Delta L_t}{\frac{1}{2}\rho(U_0^2)S} = -\eta\frac{S_t}{S}C_{L_{\alpha_t}}\frac{ql_t}{U_0}
\end{aligned}
$$

$$
\begin{aligned}
C_{Z_q} \equiv \left(\frac{\partial C_Z}{\partial(q\bar c/2U_0)}\right)_0 = \frac{2U_0}{\bar c}\left(\frac{\partial C_Z}{\partial q}\right)_0 &= -\eta\frac{2U_0}{\bar c}\frac{l_t}{U_0}\frac{S_t}{S}C_{L_{\alpha_t}} \\
&= -2\eta V_H C_{L_{\alpha_t}}
\end{aligned}
$$

- Can also show that

$$
C_{M_q} = C_{Z_q}\frac{l_t}{\bar c}
$$

Approximate Aircraft Dynamic Models

- It is often good to develop simpler models of the full set of aircraft dynamics.
  - Provides insights on the role of the aerodynamic parameters on the frequency and damping of the two modes.
  - Useful for the control design work as well
- Basic approach is to recognize that the modes have very separate sets of states that participate in the response.
  - **Short Period** – primarily θ and w in the same phase. The u and q response is very small.
  - **Phugoid** – primarily θ and u, and θ lags by about 90°. The w and q response is very small.
- Full equations from before:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\Theta_0 \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} & \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{[M_q+(Z_q+mU_0)\Gamma]}{I_{yy}} & -\frac{mg\sin\Theta_0\Gamma}{I_{yy}} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

- For the **Short Period** approximation,

  1. Since $u \approx 0$ in this mode, then $\dot{u} \approx 0$ and can eliminate the X-force equation.

$$\begin{bmatrix} \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c} \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{[M_q+(Z_q+mU_0)\Gamma]}{I_{yy}} & -\frac{mg\sin\Theta_0\Gamma}{I_{yy}} \\ 0 & 1 & 0 \end{array} \right] \begin{bmatrix} w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

  2. Typically find that $Z_{\dot{w}} \ll m$ and $Z_q \ll mU_0$. Check for 747:
     – $Z_{\dot{w}} = 1909 \ll m = 2.8866 \times 10^5$
     – $Z_q = 4.5 \times 10^5 \ll mU_0 = 6.8 \times 10^7$

$$\Gamma = \frac{M_{\dot{w}}}{m - Z_{\dot{w}}} \Rightarrow \Gamma \approx \frac{M_{\dot{w}}}{m}$$

$$\begin{bmatrix} \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c} \frac{Z_w}{m} & U_0 & -g\sin\Theta_0 \\ \frac{\left[M_w+Z_w\frac{M_{\dot{w}}}{m}\right]}{I_{yy}} & \frac{\left[M_q+(mU_0)\frac{M_{\dot{w}}}{m}\right]}{I_{yy}} & -\frac{mg\sin\Theta_0}{I_{yy}}\frac{M_{\dot{w}}}{m} \\ 0 & 1 & 0 \end{array} \right] \begin{bmatrix} w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

  3. Set $\Theta_0 = 0$ and remove θ from the model (it can be derived from q)

- With these approximations, the longitudinal dynamics reduce to

$$\dot{x}_{sp} = A_{sp}x_{sp} + B_{sp}\delta_e$$

  where δ_e is the elevator input, and

$$x_{sp} = \begin{bmatrix} w \\ q \end{bmatrix} \quad , \quad A_{sp} = \begin{bmatrix} Z_w/m & U_0 \\ I_{yy}^{-1}\left(M_w + M_{\dot{w}}Z_w/m\right) & I_{yy}^{-1}\left(M_q + M_{\dot{w}}U_0\right) \end{bmatrix}$$

$$B_{sp} = \begin{bmatrix} Z_{\delta_e}/m \\ I_{yy}^{-1}\left(M_{\delta_e} + M_{\dot{w}}Z_{\delta_e}/m\right) \end{bmatrix}$$

- Characteristic equation for this system: $s^2 + 2\zeta_{sp}\omega_{sp}s + \omega_{sp}^2 = 0$, where the **full** approximation gives:

$$2\zeta_{sp}\omega_{sp} = -\left(\frac{Z_w}{m} + \frac{M_q}{I_{yy}} + \frac{M_{\dot{w}}}{I_{yy}}U_0\right)$$

$$\omega_{sp}^2 = \frac{Z_w M_q}{m I_{yy}} - \frac{U_0 M_w}{I_{yy}}$$

- Given approximate magnitude of the derivatives for a typical aircraft, can develop a **coarse** approximate:

$$\left.\begin{array}{r} 2\zeta_{sp}\omega_{sp} \approx -\frac{M_q}{I_{yy}} \\ \omega_{sp}^2 \approx -\frac{U_0 M_w}{I_{yy}} \end{array}\right\} \rightarrow \begin{array}{l} \zeta_{sp} \approx -\frac{M_q}{2}\sqrt{\frac{-1}{U_0 M_w I_{yy}}} \\ \omega_{sp} \approx \sqrt{\frac{-U_0 M_w}{I_{yy}}} \end{array}$$

- Numerical values for 747

	Frequency rad/sec	Damping
Full model	0.962	0.387
Full Approximate	0.963	0.385
Coarse Approximate	0.906	0.187

Both approximations give the frequency well, but full approximation gives a much better damping estimate

- Approximations showed that short period mode frequency is determined by M_w – measure of the *aerodynamic stiffness in pitch.*
  - Sign of M_w negative if *cg* sufficient far forward – changes sign (mode goes unstable) when *cg* at the *stick fixed neutral point*. Follows from discussion of C_{M_α} (see 2–11)

- For the Phugoid approximation, start again with:

$$\begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g\cos\Theta_0 \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & \frac{-mg\sin\Theta_0}{m-Z_{\dot{w}}} \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} & \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{\left[M_q+(Z_q+mU_0)\Gamma\right]}{I_{yy}} & -\frac{mg\sin\Theta_0\Gamma}{I_{yy}} \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

1. Changes to w and q are very small compared to u, so we can

   – Set $\dot{w} \approx 0$ and $\dot{q} \approx 0$
   – Set $\Theta_0 = 0$

$$\begin{bmatrix} \dot{u} \\ 0 \\ 0 \\ \dot{\theta} \end{bmatrix} = \left[\begin{array}{c|c|c|c} \frac{X_u}{m} & \frac{X_w}{m} & 0 & -g \\ \frac{Z_u}{m-Z_{\dot{w}}} & \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} & 0 \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} & \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{\left[M_q+(Z_q+mU_0)\Gamma\right]}{I_{yy}} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

2. Use what is left of the Z-equation to show that with these approximations (elevator inputs)

$$\begin{bmatrix} \frac{Z_w}{m-Z_{\dot{w}}} & \frac{Z_q+mU_0}{m-Z_{\dot{w}}} \\ \frac{[M_w+Z_w\Gamma]}{I_{yy}} & \frac{\left[M_q+(Z_q+mU_0)\Gamma\right]}{I_{yy}} \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix} = -\begin{bmatrix} \frac{Z_u}{m-Z_{\dot{w}}} \\ \frac{[M_u+Z_u\Gamma]}{I_{yy}} \end{bmatrix} u - \begin{bmatrix} \frac{Z_{\delta_e}}{m-Z_{\dot{w}}} \\ \frac{\left[M_{\delta_e}+Z_{\delta_e}\Gamma\right]}{I_{yy}} \end{bmatrix} \delta_e$$

3. Use ($Z_{\dot{w}} \ll m$ so $\Gamma \approx \frac{M_{\dot{w}}}{m}$) and ($Z_q \ll mU_0$) so that:

$$\begin{bmatrix} Z_w & mU_0 \\ \left[M_w + Z_w\frac{M_{\dot{w}}}{m}\right] & [M_q + U_0 M_{\dot{w}}] \end{bmatrix} \begin{bmatrix} w \\ q \end{bmatrix}$$
$$= -\begin{bmatrix} Z_u \\ \left[M_u + Z_u\frac{M_{\dot{w}}}{m}\right] \end{bmatrix} u - \begin{bmatrix} Z_{\delta_e} \\ \left[M_{\delta_e} + Z_{\delta_e}\frac{M_{\dot{w}}}{m}\right] \end{bmatrix} \delta_e$$

4. Solve to show that

$$\begin{bmatrix} w \\ q \end{bmatrix} = \begin{bmatrix} \dfrac{mU_0M_u - Z_uM_q}{Z_wM_q - mU_0M_w} \\ \dfrac{Z_uM_w - Z_wM_u}{Z_wM_q - mU_0M_w} \end{bmatrix} u + \begin{bmatrix} \dfrac{mU_0M_{\delta_e} - Z_{\delta_e}M_q}{Z_wM_q - mU_0M_w} \\ \dfrac{Z_{\delta_e}M_w - Z_wM_{\delta_e}}{Z_wM_q - mU_0M_w} \end{bmatrix} \delta_e$$

5. Substitute into the reduced equations to get **full** approximation:

$$\begin{bmatrix} \dot{u} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{X_u}{m} + \frac{X_w}{m}\left(\frac{mU_0M_u - Z_uM_q}{Z_wM_q - mU_0M_w}\right) & -g \\ \left(\frac{Z_uM_w - Z_wM_u}{Z_wM_q - mU_0M_w}\right) & 0 \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{X_{\delta_e}}{m} + \frac{X_w}{m}\left(\frac{mU_0M_{\delta_e} - Z_{\delta_e}M_q}{Z_wM_q - mU_0M_w}\right) \\ \frac{Z_{\delta_e}M_w - Z_wM_{\delta_e}}{Z_wM_q - mU_0M_w} \end{bmatrix} \delta_e$$

6. Still a bit complicated. Typically get that
   - $|M_uZ_w| \ll |M_wZ_u|$ (1.4:4)
   - $|M_wU_0m| \gg |M_qZ_w|$ (1:0.13)
   - $|M_uX_w/M_w| \ll X_u$ small

7. With these approximations, the longitudinal dynamics reduce to the **coarse** approximation

$$\dot{x}_{ph} = A_{ph}x_{ph} + B_{ph}\delta_e$$

where δ_e is the elevator input.

And

$$x_{ph} = \begin{bmatrix} u \\ \theta \end{bmatrix} \quad A_{ph} = \begin{bmatrix} \dfrac{X_u}{m} & -g \\ \dfrac{-Z_u}{mU_0} & 0 \end{bmatrix}$$

$$B_{ph} = \begin{bmatrix} \dfrac{\left(X_{\delta_e} - \left[\frac{X_w}{M_w}\right] M_{\delta_e}\right)}{m} \\ \dfrac{\left(-Z_{\delta_e} + \left[\frac{Z_w}{M_w}\right] M_{\delta_e}\right)}{mU_0} \end{bmatrix}$$

8. Which gives

$$\begin{aligned} 2\zeta_{ph}\omega_{ph} &= -X_u/m \\ \omega_{ph}^2 &= -\frac{gZ_u}{mU_0} \end{aligned}$$

Numerical values for 747

	Frequency rad/sec	Damping
Full model	0.0673	0.0489
Full Approximate	0.0670	0.0419
Coarse Approximate	0.0611	0.0561

- Further insights: recall that

$$\left(\frac{U_0}{QS}\right)\left(\frac{\partial Z}{\partial u}\right)_0 = -\left(\frac{U_0}{QS}\right)\left(\frac{\partial L}{\partial u}\right)_0 \equiv -(C_{L_u}+2C_{L_0})$$
$$= -\frac{\mathbf{M^2}}{1-\mathbf{M^2}}C_{L_0} - 2C_{L_0} \approx -2C_{L_0}$$

so

$$Z_u \equiv \left(\frac{\partial Z}{\partial u}\right)_0 = \left(\frac{\rho U_o S}{2}\right)(-2C_{L_0}) = -\frac{2mg}{U_0}$$

- Then

$$\omega_{ph} = \sqrt{\frac{-gZ_u}{mU_0}} = \sqrt{\frac{mg^2}{mU_0^2}}$$
$$= \sqrt{2}\frac{g}{U_0}$$

which is **exactly** what Lanchester's approximation gave $\Omega \approx \sqrt{2}\frac{g}{U_0}$

- Note that

$$X_u \equiv \left(\frac{\partial X}{\partial u}\right)_0 = \left(\frac{\rho U_o S}{2}\right)(-2C_{D_0}) = -\rho U_o S C_{D_0}$$

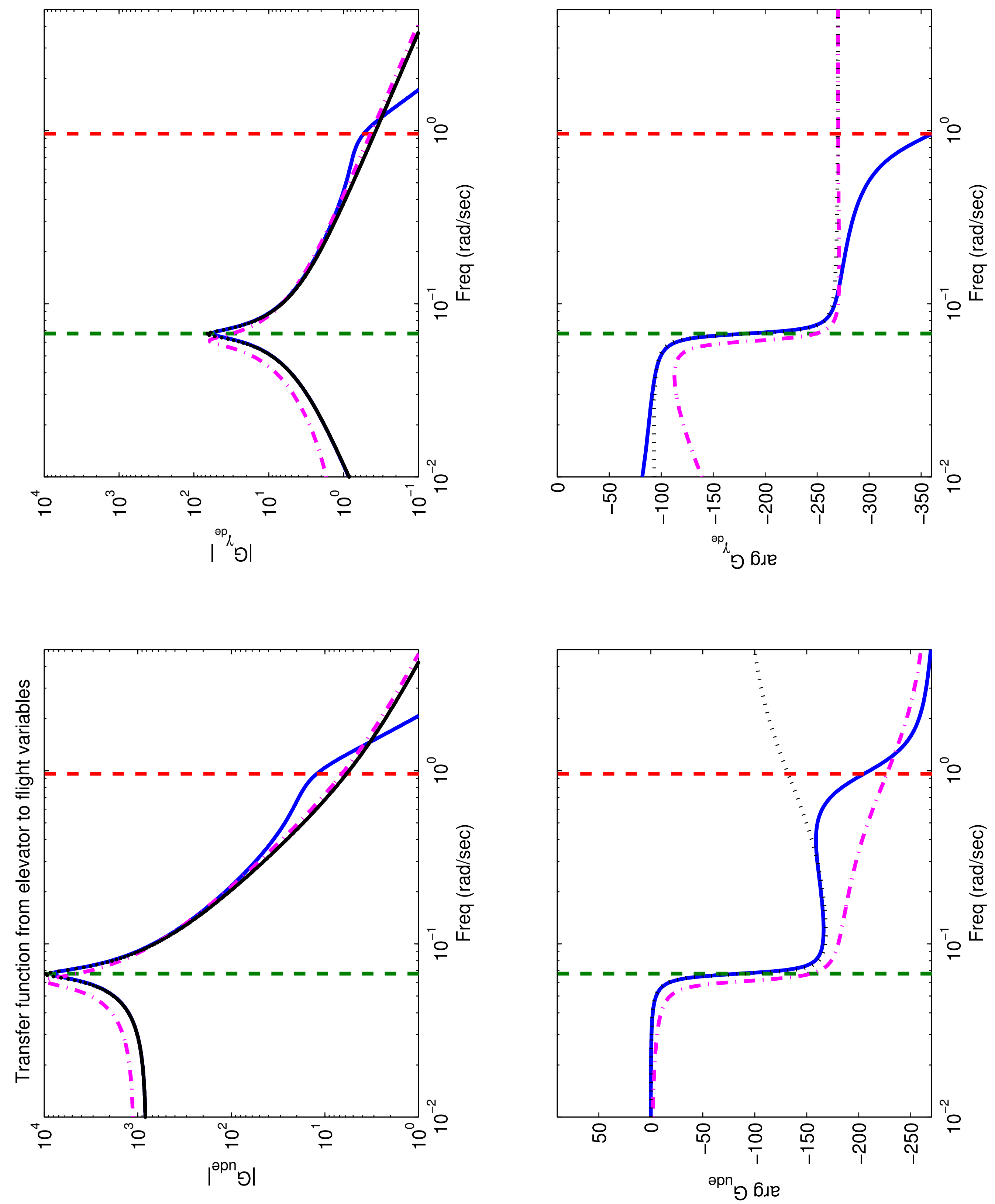
and

$$2mg = \rho U_o^2 S C_{L_0}$$

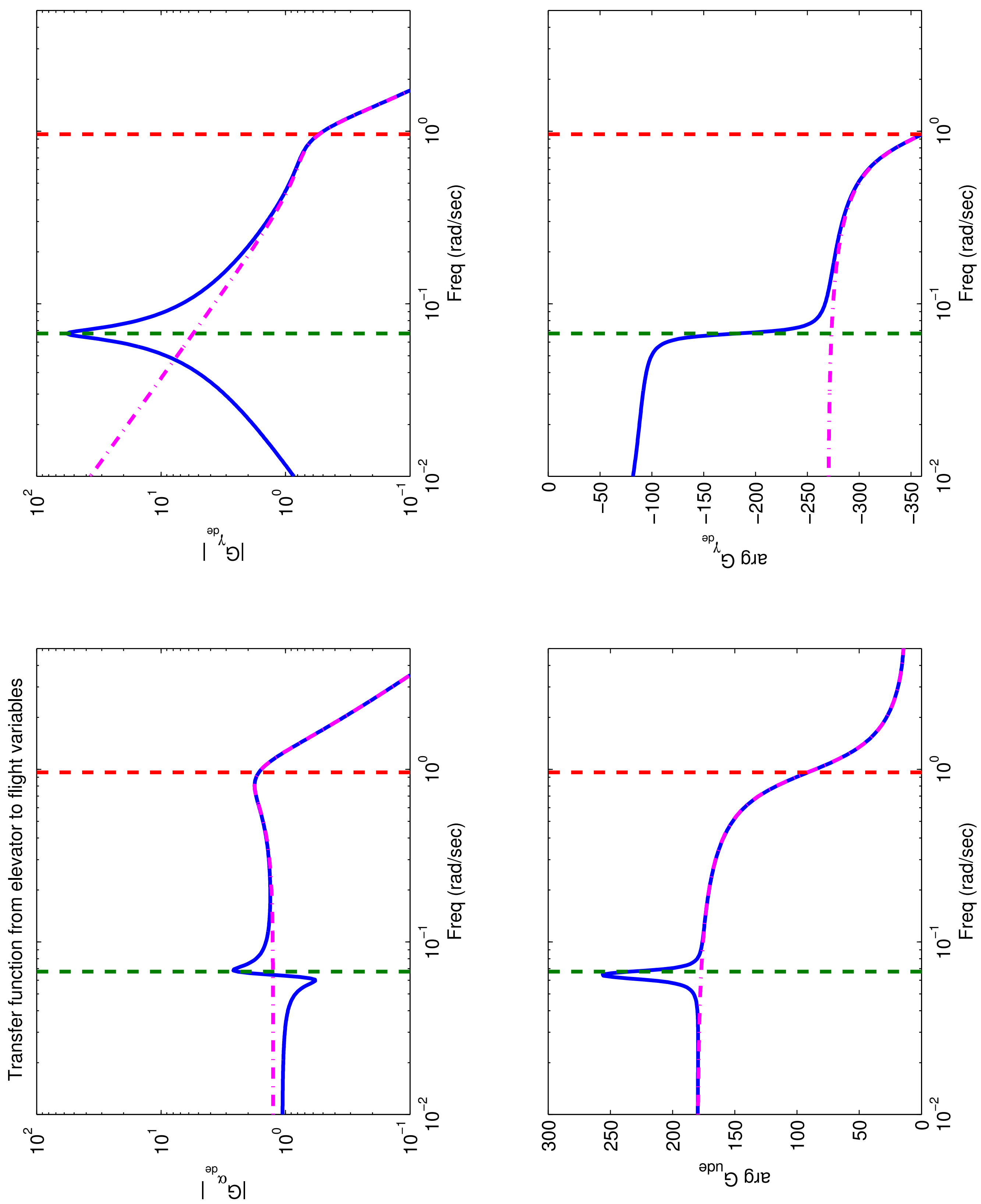
so

$$\zeta_{ph} = -\frac{X_u}{2m\omega_{ph}} = -\frac{X_u U_0}{2\sqrt{2}mg}$$
$$= \frac{1}{\sqrt{2}}\left(\frac{\rho U_o^2 S C_{D_0}}{\rho U_o^2 S C_{L_0}}\right)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{C_{D_0}}{C_{L_0}}\right)$$

so the damping ratio of the approximate phugoid mode is **inversely proportional** to the **lift to drag** ratio.



Freq Comparison from elevator (Phugoid Model) – B747 at M=0.8. **Blue**– Full model, **Black**– Full approximate model, **Magenta**– Coarse approximate model



Freq Comparison from elevator (Short Period Model) – B747 at M=0.8. **Blue**– Full model, **Magenta**– Approximate model

Summary

- Approximate longitudinal models are fairly accurate

- Indicate that the aircraft responses are mainly determined by these stability derivatives:

Property	Stability derivative
Damping of the short period	M_q
Frequency of the short period	M_w
Damping of the Phugoid	X_u
Frequency of the Phugoid	Z_u

- Given a change in α, expect changes in u as well. These will both impact the lift and drag of the aircraft, requiring that we re-trim throttle setting to maintain whatever aspects of the flight condition might have changed (other than the ones we wanted to change). We have:

$$\begin{bmatrix} \Delta L \\ \Delta D \end{bmatrix} = \begin{bmatrix} L_u & L_\alpha \\ D_u & D_\alpha \end{bmatrix} \begin{bmatrix} u \\ \Delta\alpha \end{bmatrix}$$

But to maintain $L = W$, want $\Delta L = 0$, so $u = -\frac{L_\alpha}{L_u}\Delta\alpha$

Giving $\Delta D = \left(-\frac{L_\alpha}{L_u} D_u + D_\alpha\right)\Delta\alpha$

$$\begin{aligned} C_{D_\alpha} = \frac{2C_{L_0}}{\pi e A\!\!R} C_{L_\alpha} \;&\rightarrow\; D_\alpha = QSC_{D_\alpha} \\ &\rightarrow\; L_\alpha = QSC_{L_\alpha} \\ & \quad\; D_u = \frac{QS}{U_0}(2C_{D_0}) \qquad (4-16) \\ & \quad\; L_u = \frac{QS}{U_0}(2C_{L_0}) \qquad (4-17) \end{aligned}$$

$$\begin{aligned} \Delta D &= QS\left(-\frac{C_{L_\alpha}}{2C_{L_0}/U_0}\left(\frac{2C_{D_0}}{U_0}\right) + C_{D_\alpha}\right)\Delta\alpha \\ &= \frac{QS}{C_{L_0}}\left(-C_{D_0} + \frac{2C_{L_0}^2}{\pi e A\!\!R}\right) C_{L_\alpha}\Delta\alpha \end{aligned}$$

$$\begin{aligned} \tan\Delta\gamma &= \frac{(T_0 + \Delta T) - (D_0 + \Delta D)}{L_0 + \Delta L} = \frac{-\Delta D}{L_0} \\ &= \left(\frac{C_{D_0}}{C_{L_0}} - \frac{2C_{L_0}}{\pi e A\!\!R}\right)\frac{C_{L_\alpha}\Delta\alpha}{C_{L_0}} \end{aligned}$$

For 747 (Reid 165 and Nelson 416), $A\!\!R = 7.14$, so $\pi e A\!\!R \approx 18$, $C_{L_0} = 0.654$ $C_{D_0} = 0.043$, $C_{L_\alpha} = 5.5$, for a $\Delta\alpha = -0.0185$rad (6–7) $\Delta\gamma = -0.0006$rad. This is the opposite sign to the linear simulation results, but they are both very small numbers.