# 16.333: Lecture # 6

Aircraft Longitudinal Dynamics

- Typical aircraft open-loop motions
- Longitudinal modes
- Impact of actuators
- **Linear Algebra in Action!**

# Longitudinal Dynamics

- Recall: $X$ denotes the force in the $X$-direction, and similarly for $Y$ and $Z$, then (as on 4–13)

$$X_u \equiv \left(\frac{\partial X}{\partial u}\right)_0, \ldots$$

- Longitudinal equations (see 4–13) can be rewritten as:

$$\begin{aligned} m\dot{u} &= X_u u + X_w w - mg\cos\Theta_0\theta + \Delta X^c \\ m(\dot{w} - qU_0) &= Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q - mg\sin\Theta_0\theta + \Delta Z^c \\ I_{yy}\dot{q} &= M_u u + M_w w + M_{\dot{w}}\dot{w} + M_q q + \Delta M^c \end{aligned}$$

- There is no roll/yaw motion, so $q = \dot{\theta}$.

- Control commands $\Delta X^c$, $\Delta Z^c$, and $\Delta M^c$ have not yet been specified.

- Rewrite in **state space** form as

$$\begin{bmatrix} m\dot{u} \\ (m - Z_{\dot{w}})\dot{w} \\ -M_{\dot{w}}\dot{w} + I_{yy}\dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -mg\cos\Theta_0 \\ Z_u & Z_w & Z_q + mU_0 & -mg\sin\Theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m - Z_{\dot{w}} & 0 & 0 \\ 0 & -M_{\dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} X_u & X_w & 0 & -mg\cos\Theta_0 \\ Z_u & Z_w & Z_q + mU_0 & -mg\sin\Theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix}$$

$$E\dot{\mathcal{X}} = \bar{A}\mathcal{X} + \hat{\mathbf{c}} \qquad \text{descriptor state space form}$$

$$\Rightarrow \dot{\mathcal{X}} = E^{-1}(\bar{A}\mathcal{X} + \hat{\mathbf{c}}) = A\mathcal{X} + \mathbf{c}$$

- Write out in state space form:

$$A = \left[\begin{array}{c|c|c|c} \dfrac{X_u}{m} & \dfrac{X_w}{m} & 0 & -g\cos\Theta_0 \\ \\ \dfrac{Z_u}{m - Z_{\dot{w}}} & \dfrac{Z_w}{m - Z_{\dot{w}}} & \dfrac{Z_q + mU_0}{m - Z_{\dot{w}}} & \dfrac{-mg\sin\Theta_0}{m - Z_{\dot{w}}} \\ \\ I_{yy}^{-1}\left[M_u + Z_u\Gamma\right] & I_{yy}^{-1}\left[M_w + Z_w\Gamma\right] & I_{yy}^{-1}\left[M_q + (Z_q + mU_0)\Gamma\right] & -I_{yy}^{-1}mg\sin\Theta_0\Gamma \\ \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\Gamma = \frac{M_{\dot{w}}}{m - Z_{\dot{w}}}$$

- Note: slight savings if we defined symbols to embed the mass/inertia $\hat{X}_u = X_u/m$, $\hat{Z}_u = Z_u/m$, and $\hat{M}_q = M_q/I_{yy}$ then A matrix collapses to:

$$\hat{A} = \left[\begin{array}{c|c|c|c} \hat{X}_u & \hat{X}_w & 0 & -g\cos\Theta_0 \\ \\ \dfrac{\hat{Z}_u}{1 - \hat{Z}_{\dot{w}}} & \dfrac{\hat{Z}_w}{1 - \hat{Z}_{\dot{w}}} & \dfrac{\hat{Z}_q + U_0}{1 - \hat{Z}_{\dot{w}}} & \dfrac{-g\sin\Theta_0}{1 - \hat{Z}_{\dot{w}}} \\ \\ \left[\hat{M}_u + \hat{Z}_u\hat{\Gamma}\right] & \left[\hat{M}_w + \hat{Z}_w\hat{\Gamma}\right] & \left[\hat{M}_q + (\hat{Z}_q + U_0)\hat{\Gamma}\right] & -g\sin\Theta_0\hat{\Gamma} \\ \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\hat{\Gamma} = \frac{\hat{M}_{\dot{w}}}{1 - \hat{Z}_{\dot{w}}}$$

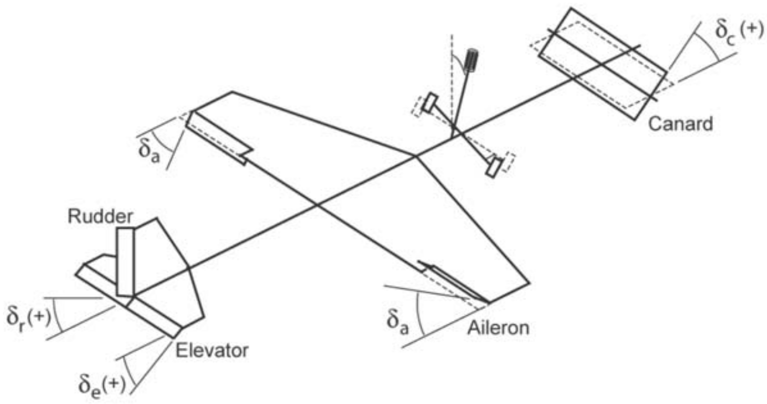
- Check the notation that is being used very carefully

- To figure out the $\mathbf{c}$ vector, we have to say a little more about how the control inputs are applied to the system.

# Longitudinal Actuators

- Primary actuators in longitudinal direction are the elevators and thrust.
  - Clearly the thrusters/elevators play a key role in defining the steady-state/equilibrium flight condition
  - Now interested in determining how they also influence the aircraft motion about this equilibrium condition

$$\text{deflect elevator} \quad \rightarrow \quad u(t), w(t), q(t), \ldots$$




- Recall that we defined $\Delta X^c$ as the perturbation in the total force in the $X$ direction as a result of the actuator commands
  - Force change due to an actuator deflection from trim

- Expand these aerodynamic terms using same perturbation approach

$$\Delta X^c = X_{\delta_e}\delta_e + X_{\delta_p}\delta_p$$

  - $\delta_e$ is the deflection of the elevator from trim (down positive)
  - $\delta_p$ change in thrust
  - $X_{\delta_e}$ and $X_{\delta_p}$ are the **control stability derivatives**

- Now we have that

$$\mathbf{c} \;=\; E^{-1}\begin{bmatrix} \Delta X^c \\ \Delta Z^c \\ \Delta M^c \\ 0 \end{bmatrix} = E^{-1}\begin{bmatrix} X_{\delta_e} & X_{\delta_p} \\ Z_{\delta_e} & Z_{\delta_p} \\ M_{\delta_e} & M_{\delta_p} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \delta_e \\ \delta_p \end{bmatrix} = Bu$$

- For the longitudinal case

$$B = \left[\begin{array}{c|c} \dfrac{X_{\delta_e}}{m} & \dfrac{X_{\delta_p}}{m} \\ \dfrac{Z_{\delta_e}}{m - Z_{\dot{w}}} & \dfrac{Z_{\delta_p}}{m - Z_{\dot{w}}} \\ I_{yy}^{-1}\left[M_{\delta_e} + Z_{\delta_e}\Gamma\right] & I_{yy}^{-1}\left[M_{\delta_p} + Z_{\delta_p}\Gamma\right] \\ 0 & 0 \end{array}\right]$$

- Typical values for the B747

$$\begin{array}{ll} X_{\delta_e} = -16.54 & X_{\delta_p} = 0.3mg = 849528 \\ Z_{\delta_e} = -1.58 \cdot 10^6 & Z_{\delta_p} \approx 0 \\ M_{\delta_e} = -5.2 \cdot 10^7 & M_{\delta_p} \approx 0 \end{array}$$

- Aircraft response $y = G(s)u$

$$\begin{array}{l} \dot{\mathcal{X}} = A\mathcal{X} + Bu \;\rightarrow\; G(s) = C(sI - A)^{-1}B \\ y = C\mathcal{X} \end{array}$$

- We now have the means to modify the dynamics of the system, but first let's figure out what $\delta_e$ and $\delta_p$ really do.

# Longitudinal Response

- **Final response** to a step input $u = \hat{u}/s$, $y = G(s)u$, use the **FVT**

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\left(G(s)\frac{\hat{u}}{s}\right)$$

$$\Rightarrow \lim_{t\to\infty} y(t) = G(0)\hat{u} = -(CA^{-1}B)\hat{u}$$

- **Initial response** to a step input, use the **IVT**

$$y(0^+) = \lim_{s\to\infty} s\left(G(s)\frac{\hat{u}}{s}\right) = \lim_{s\to\infty} G(s)\hat{u}$$

  – For your system, $G(s) = C(sI - A)^{-1}B + D$, but $D \equiv 0$, so

$$\lim_{s\to\infty} G(s) \to 0$$

  – **Note: there is NO immediate change** in the output of the motion variables in response to an elevator input $\Rightarrow y(0^+) = 0$

- Consider the *rate of change* of these variables $\dot{\mathbf{y}}(\mathbf{0}^+)$

$$\dot{y}(t) = C\dot{\mathcal{X}} = CA\mathcal{X} + CBu$$

and normally have that $CB \neq 0$. Repeat process above[1] to show that $\dot{y}(0^+) = CB\hat{u}$, and since $C \equiv I$,

$$\dot{y}(0^+) = B\hat{u}$$

- Looks good. Now compare with numerical values computed in Matlab. Plot $u$, $\alpha$, and flight path angle $\gamma = \theta - \alpha$ (since $\Theta_0 = \gamma_0 = 0$ – see picture on 4–8)

[1] Note that $C(sI - A)^{-1}B + D = D + \frac{CB}{s} + \frac{CA^{-1}B}{s^2} + \ldots$

## Elevator ($1^\circ$ elevator down – stick forward)

- See very rapid response that decays quickly (mostly in the first 10 seconds of the $\alpha$ response)
- Also see a very lightly damped long period response (mostly $u$, some $\gamma$, and very little $\alpha$). Settles in $>$600 secs
- Predicted **steady state** values from code:

| | | | |
|---|---|---|---|
| 14.1429 | m/s | $u$ | (speeds up) |
| -0.0185 | rad | $\alpha$ | (slight reduction in AOA) |
| -0.0000 | rad/s | $q$ | |
| -0.0161 | rad | $\theta$ | |
| 0.0024 | rad | $\gamma$ | |

  – Predictions appear to agree well with the numerical results.
  – **Primary result** is a **slightly lower angle of attack and a higher speed**

- Predicted **initial rates** of the output values from code:

| | | |
|---|---|---|
| -0.0001 | m/s$^2$ | $\dot{u}$ |
| -0.0233 | rad/s | $\dot{\alpha}$ |
| -1.1569 | rad/s$^2$ | $\dot{q}$ |
| 0.0000 | rad/s | $\dot{\theta}$ |
| 0.0233 | rad/s | $\dot{\gamma}$ |

  – All outputs are zero at $t = 0^+$, but see rapid changes in $\alpha$ and $q$.
  – Changes in $u$ and $\gamma$ (also a function of $\theta$) are much more gradual – not as easy to see this aspect of the prediction

- **Initial impact** Change in $\alpha$ and $q$ (pitches aircraft)
- **Long term impact** Change in $u$ (determines speed at new equilibrium condition)

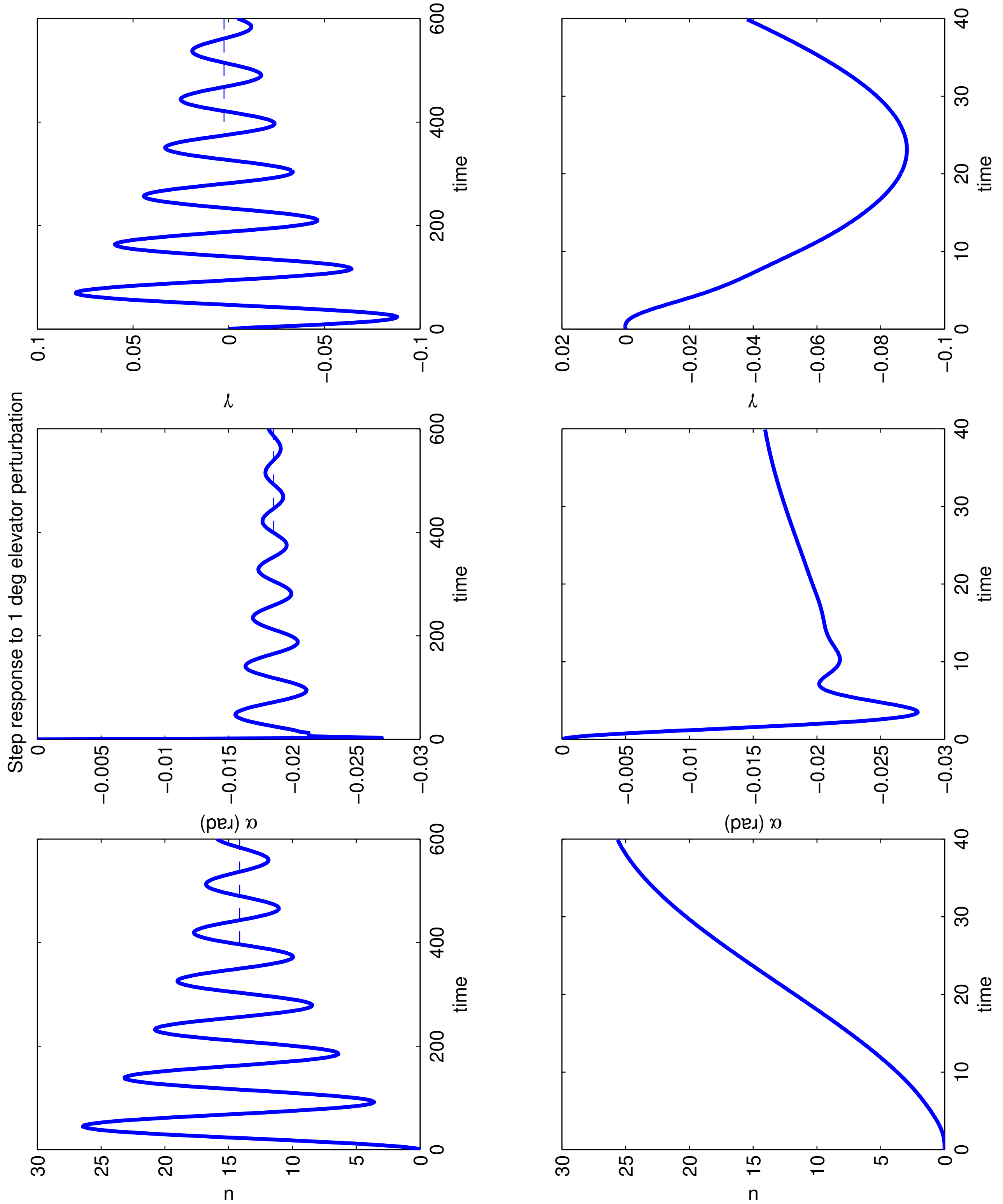



Figure 1: Step Response to 1 deg elevator perturbation – B747 at M=0.8

## Thrust (1/6 input)

- Motion now dominated by the lightly damped long period response
- Short period motion barely noticeable at beginning.
- Predicted **steady state** values from code:

| | | |
|---|---|---|
| 0 | m/s | $u$ |
| 0 | rad | $\alpha$ |
| 0 | rad/s | $q$ |
| 0.05 | rad | $\theta$ |
| 0.05 | rad | $\gamma$ |

  – Predictions appear to agree well with the simulations.
  – **Primary result – now climbing with a flight path angle of 0.05 rad at the same speed we were going before.**

- Predicted **initial rates** of the output values from code:

| | | |
|---|---|---|
| 2.9430 | m/s$^2$ | $\dot{u}$ |
| 0 | rad/s | $\dot{\alpha}$ |
| 0 | rad/s$^2$ | $\dot{q}$ |
| 0 | rad/s | $\dot{\theta}$ |
| 0 | rad/s | $\dot{\gamma}$ |

  – Changes to $\alpha$ are very small, and $\gamma$ response initially flat.
  – Increase power, and the aircraft initially speeds up

- **Initial impact** Change in $u$ (accelerates aircraft)
- **Long term impact** Change in $\gamma$ (determines climb rate)

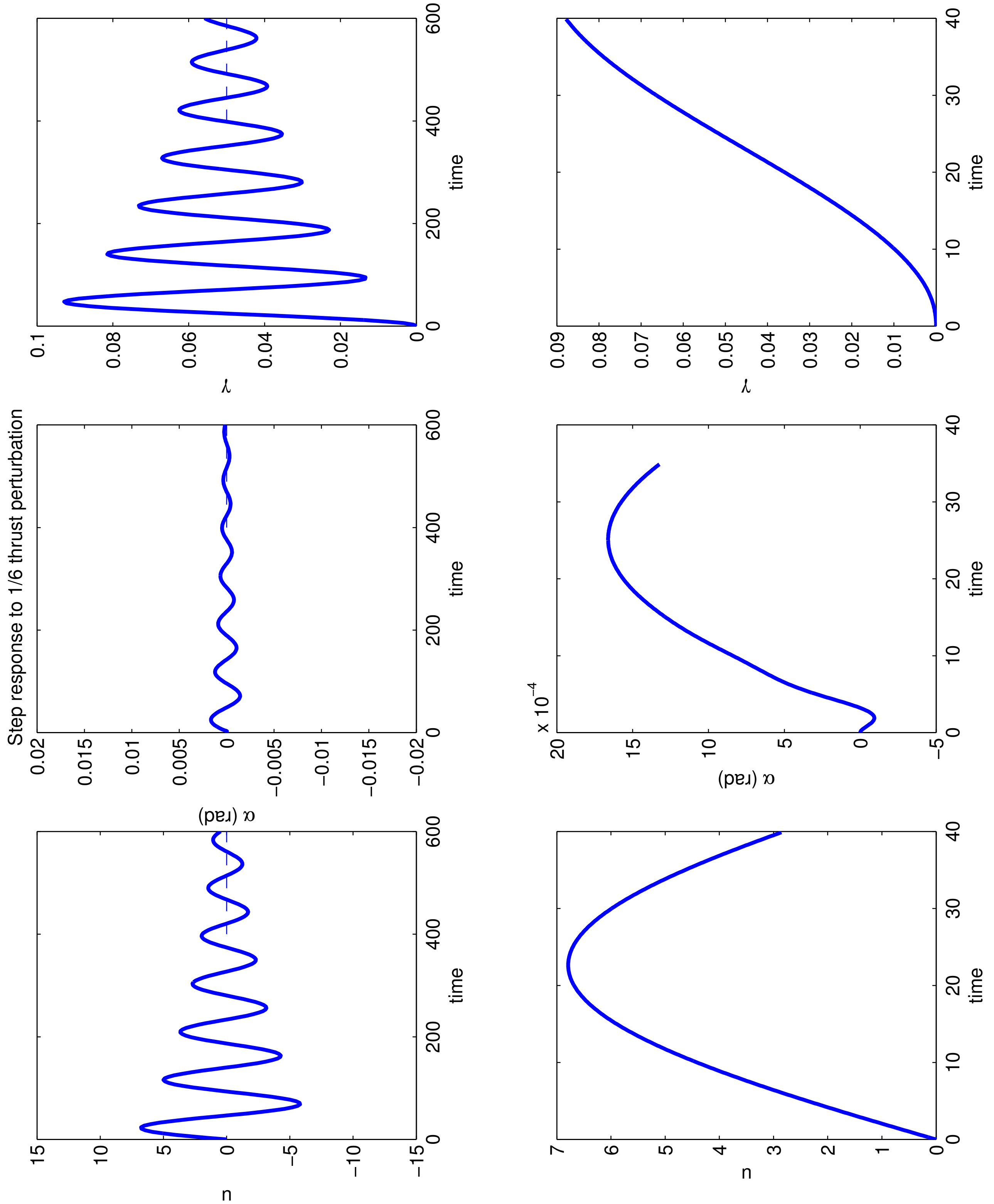



Figure 2: Step Response to 1/6 thrust perturbation – B747 at M=0.8

# Frequency Domain Response

- Plot and inspect transfer functions from $\delta_e$ and $\delta_p$ to $u$, $w$, and $\gamma$
  - See following pages

- **From elevator:**
  - Huge response at the phugoid mode for both $u$ and $\gamma$ (very lightly damped)
  - Short period mode less pronounced
  - Response falls off very rapidly
  - Response to $w$ shows a pole/zero cancelation (almost) of the phugoid mode. So the magnitude level is essentially constant out to the frequency of the short period mode

    Why would we expect that?

- **From thrust:**
  - Phugoid peaks present, but short period mode is very weak (not in $u$, low in $\gamma$, $w$). $\Rightarrow$ entirely consistent with the step response.
  - Thrust controls speed (initially), so we would expect to see a large response associated with the phugoid mode (speed variations are a key component of this mode)

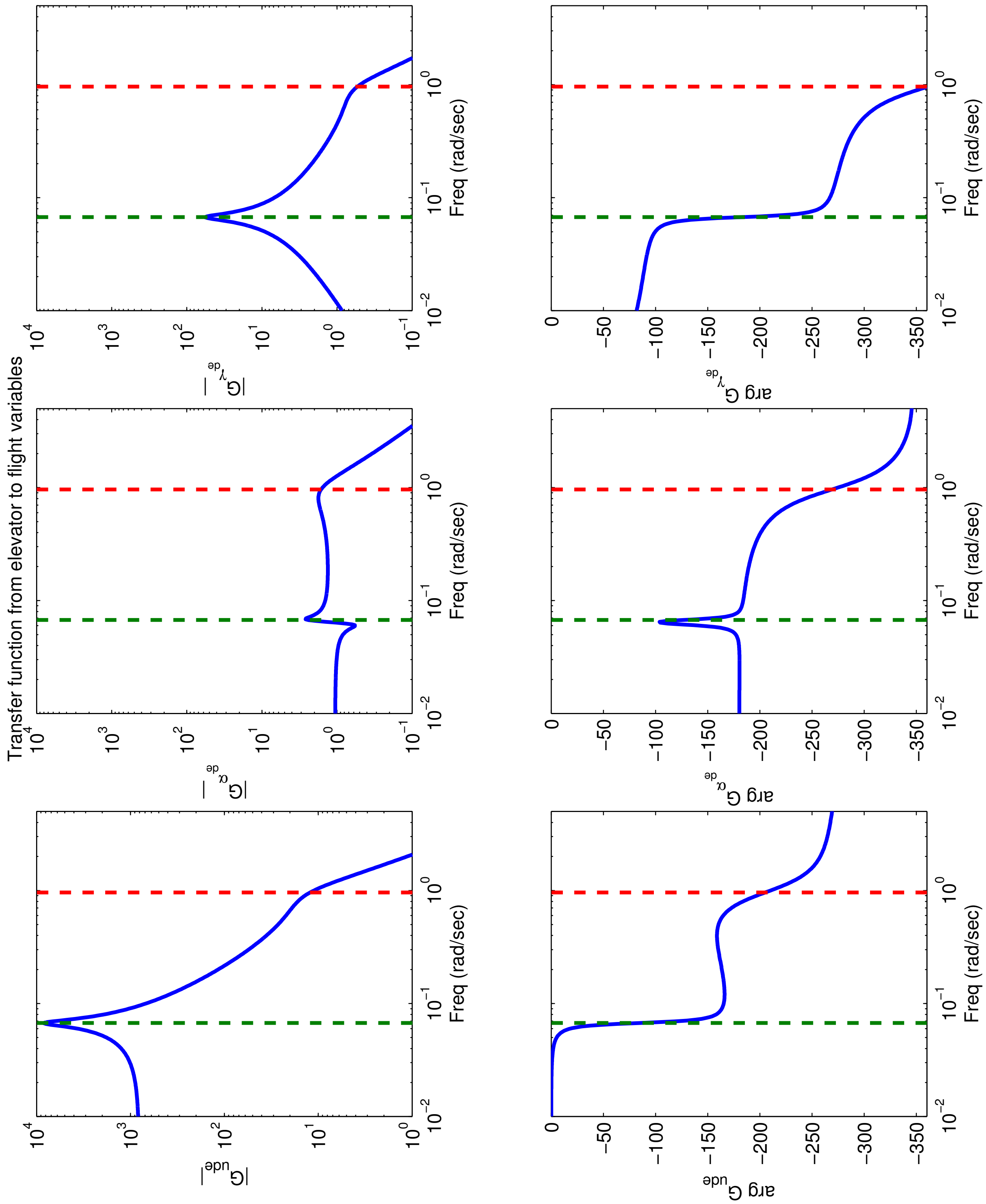



Figure 3: TF's from elevator to flight variables – B747 at M=0.8

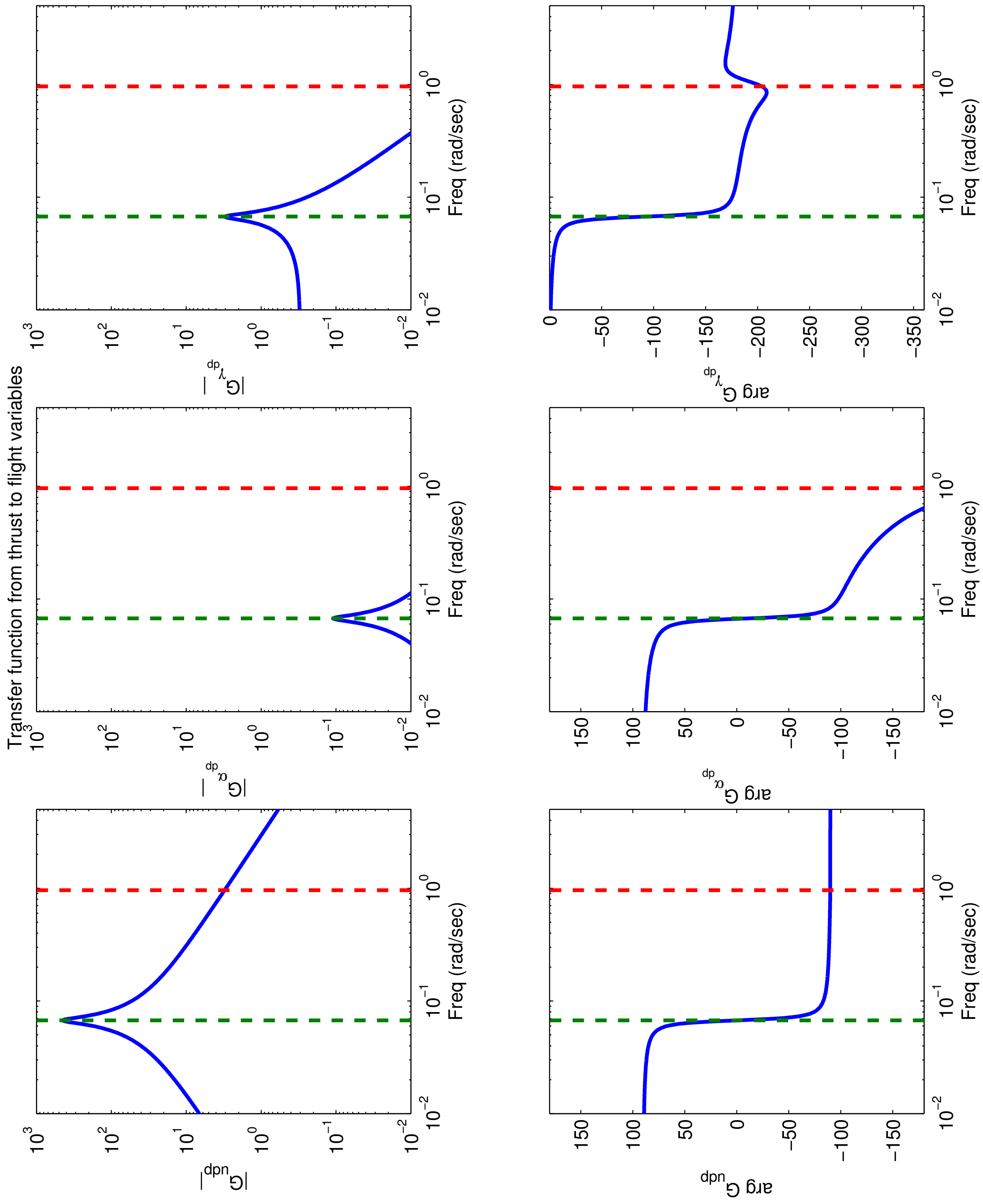



Figure 4: TF's from thrust to flight variables– B747 at M=0.8

- **Summary:**
  - **To increase equilibrium climb rate, add power.**
  - **To increase equilibrium speed, increase $\delta_e$ (move elevator further down).**
  - Transient (initial) effects are the opposite **and tend to be more consistent with what you would intuitively expect to occur**

# Modal Behavior

- Analyze model of vehicle dynamics to quantify the responses seen.
  - Homogeneous dynamics of the form $\dot{X} = AX$, so the response is

$$X(t) = e^{At}X(0) \text{ – a matrix exponential.}$$

- To simplify the investigation of the system response, find the **modes** of the system using the *eigenvalues* and *eigenvectors*
  - $\lambda$ is an **eigenvalue** of $A$ if $\det(\lambda I - A) = 0$ which is true iff there exists a nonzero $v$ (**eigenvector**) for which

$$(\lambda I - A)v = 0 \quad \Rightarrow \quad Av = \lambda v$$

  - If A $(n \times n)$, typically get $n$ eigenvalues/eigenvectors $Av_i = \lambda_i v_i$
  - Assuming that eigenvectors are **linearly independent**, can form

$$A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$AT = T\Lambda$$

$$\Rightarrow T^{-1}AT = \Lambda \quad , \quad A = T\Lambda T^{-1}$$

  - Given that $e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots$, and that $A = T\Lambda T^{-1}$, then it is easy to show that

$$X(t) = e^{At}X(0) = Te^{\Lambda t}T^{-1}X(0) = \sum_{i=1}^{n} v_i e^{\lambda_i t}\beta_i$$

  - State solution is a linear combination of **system modes** $v_i e^{\lambda_i t}$

$e^{\lambda_i t}$ – determines **nature** of the time response

$v_i$ – gives extent to which each state **participates** in that mode

$\beta_i$ – determines extent to which initial condition **excites** the mode

- The total behavior of the system can be found from the system modes
- Consider numerical example of B747

$$A = \begin{bmatrix} -0.0069 & 0.0139 & 0 & -9.8100 \\ -0.0905 & -0.3149 & 235.8928 & 0 \\ 0.0004 & -0.0034 & -0.4282 & 0 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix}$$

which gives two sets of complex eigenvalues

$$\lambda = -0.3717 \pm 0.8869\ \mathbf{i}, \quad \omega = 0.962, \quad \zeta = 0.387, \qquad \text{short period}$$

$$\lambda = -0.0033 \pm 0.0672\ \mathbf{i}, \quad \omega = 0.067, \quad \zeta = 0.049, \quad \text{Phugoid - long period}$$

  – **Result is consistent with step response** - heavily damped fast response, and a lightly damped slow one.

- To understand eigenvectors, must do some normalization (scales each element appropriately so that we can compare relative sizes)
  – $\hat{u} = u/U_0$, $\alpha = w/U_0$, $\hat{q} = q/(2U_0/\bar{c})$
  – Then divide through so that $\theta \equiv 1$

| | Short Period | Phugoid |
|---|---|---|
| $\hat{u}$ | $0.0156 + 0.0244\ \mathbf{i}$ | $-0.0254 + 0.6165\ \mathbf{i}$ |
| $\alpha$ | $1.0202 + 0.3553\ \mathbf{i}$ | $0.0045 + 0.0356\ \mathbf{i}$ |
| $\hat{q}$ | $-0.0066 + 0.0156\ \mathbf{i}$ | $-0.0001 + 0.0012\ \mathbf{i}$ |
| $\theta$ | $1.0000$ | $1.0000$ |

- **Short Period** – primarily $\theta$ and $\alpha = \hat{w}$ in the same phase. The $\hat{u}$ and $\hat{q}$ response is very small.
- **Phugoid** – primarily $\theta$ and $\hat{u}$, and $\theta$ lags by about 90°. The $\alpha$ and $\hat{q}$ response is very small.
- Dominant behavior agrees with time step responses – note how initial conditions were formed.

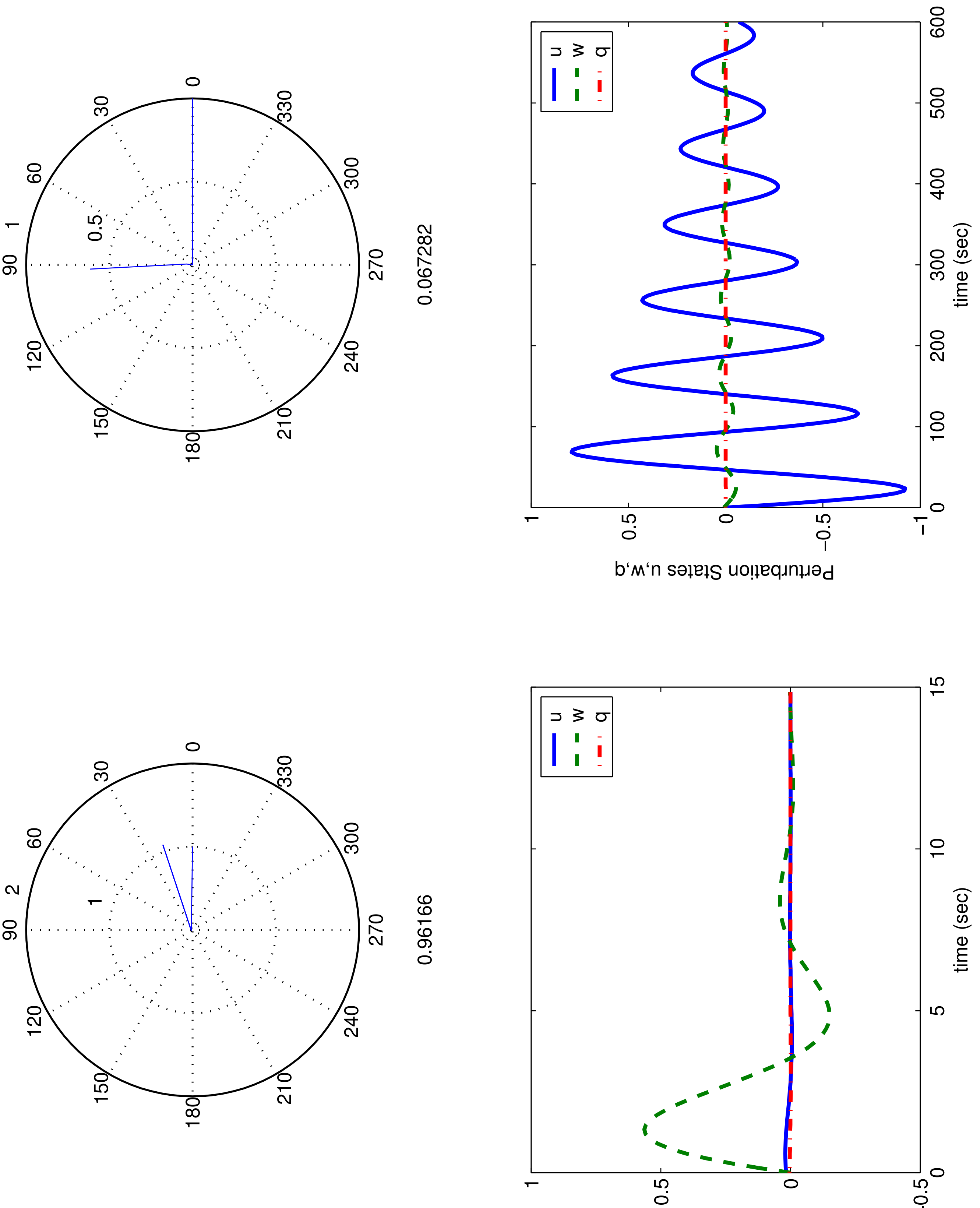



Figure 5: Mode Response – B747 at M=0.8

- Relative motion between aircraft and an observer flying at a constant speed $U_0 t$

(Image removed for copyright considerations.)

- Motion of perturbed aircraft with respect to an unperturbed one
- Note phasing of the forward velocity $\dot{x}_e$ with respect to altitude $z_e$
  - aircraft faster than observer at the bottom, slower at the top
  - The aircraft speeds up and slows down – leads and lags the observer.

- Consistent with flight path?
- Consistent with Lanchester's approximation on 4–1?

# Summary

- Two primary longitudinal modes: phugoid and short-period
  - Have versions from the full model – but can develop good approximations that help identify the aerodynamic features that determine the mode frequencies and damping

  Impact of the various actuators clarified:
  - Short time-scale
  - Long time-scale

# Matrix Diagonalization

- Suppose $A$ is diagonizable with independent eigenvectors

$$V = [v_1, \ldots, v_n]$$

  – use similarity transformations to diagonalize dynamics matrix

$$\begin{aligned} \dot{x} &= Ax \Rightarrow \dot{x}_d = A_d x_d \\ V^{-1}AV &= \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \triangleq \Lambda = A_d \end{aligned}$$

  – Corresponds to change of state from $x$ to $x_d = V^{-1}x$

- System response given by $e^{At}$, look at power series expansion

$$\begin{aligned} At &= V\Lambda t V^{-1} \\ (At)^2 &= (V\Lambda t V^{-1})V\Lambda t V^{-1} = V\Lambda^2 t^2 V^{-1} \\ \Rightarrow (At)^n &= V\Lambda^n t^n V^{-1} \end{aligned}$$

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}(At)^2 + \ldots \\ &= V\left\{ I + \Lambda + \frac{1}{2}\Lambda^2 t^2 + \ldots \right\} V^{-1} \\ &= Ve^{\Lambda t}V^{-1} = V \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} V^{-1} \end{aligned}$$

- Taking Laplace transform,

$$(sI - A)^{-1} = V \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} V^{-1}$$
$$= \sum_{i=1}^{n} \frac{R_i}{s-\lambda_i}$$

where the residue $R_i = v_i w_i^T$, and we define

$$V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} \quad , V^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

- Note that the $w_i$ are the left eigenvectors of $A$ associated with the right eigenvectors $v_i$

$$AV = V \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \Rightarrow V^{-1}A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} V^{-1}$$

$$\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$$

where $w_i^T A = \lambda_i w_i^T$

- So, if $\dot{x} = Ax$, the time domain solution is given by

$$
\begin{aligned}
x(t) &= \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T x(0) \qquad \text{dyad} \\
x(t) &= \sum_{i=1}^{n} [w_i^T x(0)] e^{\lambda_i t} v_i
\end{aligned}
$$

- The part of the solution $v_i e^{\lambda_i t}$ is called a **mode** of a system
  - solution is a weighted sum of the system modes
  - weights depend on the components of $x(0)$ along $w_i$

- Can now give dynamics interpretation of left and right eigenvectors:

$$
Av_i = \lambda_i v_i \quad , w_i A = \lambda_i w_i \quad , w_i^T v_j = \delta_{ij}
$$

so if $x(0) = v_i$, then

$$
\begin{aligned}
x(t) &= \sum_{i=1}^{n} (w_i^T x(0)) e^{\lambda_i t} v_i \\
&= e^{\lambda_i t} v_i
\end{aligned}
$$

$\Rightarrow$ so **right** eigenvectors are initial conditions that result in relatively simple motions $x(t)$.

**With no external inputs, if initial condition only disturbs one mode, then the response consists of only that mode for all time.**

- If $A$ has complex conjugate eigenvalues, the process is similar but a little more complicated.
- Consider a 2x2 case with $A$ having eigenvalues $a \pm b\mathbf{i}$ and associated eigenvectors $e_1$, $e_2$, with $e_2 = \bar{e}_1$. Then

$$\begin{aligned} A &= \left[\, e_1 \,\middle|\, e_2 \,\right] \begin{bmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, e_2 \,\right]^{-1} \\ &= \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right] \begin{bmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right]^{-1} \equiv TDT^{-1} \end{aligned}$$

- Now use the transformation matrix

$$M = 0.5 \begin{bmatrix} 1 & -\mathbf{i} \\ 1 & \mathbf{i} \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 1 \\ \mathbf{i} & -\mathbf{i} \end{bmatrix}$$

- Then it follows that

$$\begin{aligned} A &= TDT^{-1} = (TM)(M^{-1}DM)(M^{-1}T^{-1}) \\ &= (TM)(M^{-1}DM)(TM)^{-1} \end{aligned}$$

which has the nice structure:

$$A = \left[\, \mathtt{Re}(e_1) \,\middle|\, \mathtt{Im}(e_1) \,\right] \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \left[\, \mathtt{Re}(e_1) \,\middle|\, \mathtt{Im}(e_1) \,\right]^{-1}$$

where all the matrices are real.

- With complex roots, the diagonalization is to a block diagonal form.

- For this case we have that

$$e^{At} = \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1}$$

- Note that $\left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1}$ is the matrix that inverts $\left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]$

$$\left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1} \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- So for an initial condition to excite just this mode, can pick $x(0) = [\texttt{Re}(e_1)]$, or $x(0) = [\texttt{Im}(e_1)]$ or a linear combination.

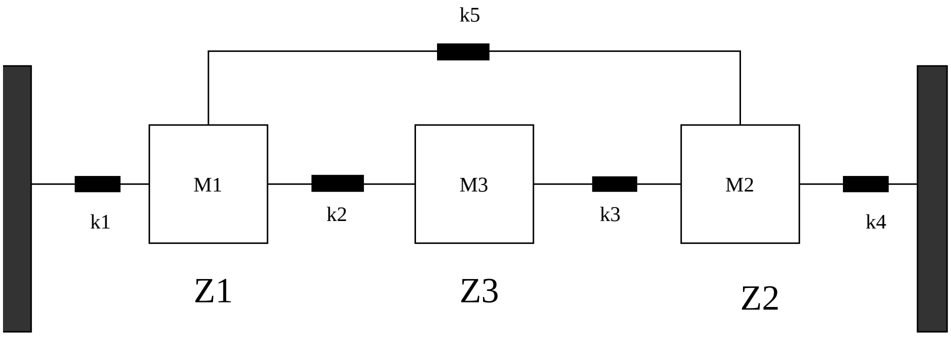
- Example $x(0) = [\texttt{Re}(e_1)]$

$$\begin{aligned} x(t) &= e^{At}x(0) = \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \cdot \\ &\qquad \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1} [\texttt{Re}(e_1)] \\ &= \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= e^{at} \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] \begin{bmatrix} \cos(bt) \\ -\sin(bt) \end{bmatrix} \\ &= e^{at} \left( \texttt{Re}(e_1) \cos(bt) - \texttt{Im}(e_1) \sin(bt) \right) \end{aligned}$$

which would ensure that only this mode is excited in the response

# Example: Spring Mass System

- Classic example: spring mass system consider simple case first: $m_i = 1$, and $k_i = 1$




$$\begin{aligned} x &= \begin{bmatrix} z_1 & z_2 & z_3 & \dot{z}_1 & \dot{z}_2 & \dot{z}_3 \end{bmatrix} \\ A &= \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \quad M = \mathrm{diag}(m_i) \\ K &= \begin{bmatrix} k_1+k_2+k_5 & -k_5 & -k_2 \\ -k_5 & k_3+k_4+k_5 & -k_3 \\ -k_2 & -k_3 & k_2+k_3 \end{bmatrix} \end{aligned}$$

- Eigenvalues and eigenvectors of the undamped system

$$\lambda_1 = \pm 0.77\mathbf{i} \quad \lambda_2 = \pm 1.85\mathbf{i} \quad \lambda_3 = \pm 2.00\mathbf{i}$$

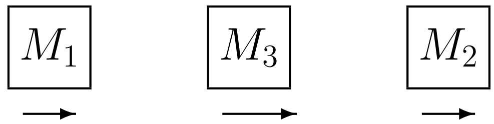
| $v_1$ | $v_2$ | $v_3$ |
|---|---|---|
| 1.00 | 1.00 | 1.00 |
| 1.00 | 1.00 | $-1.00$ |
| 1.41 | $-1.41$ | 0.00 |
| $\pm 0.77\mathbf{i}$ | $\pm 1.85\mathbf{i}$ | $\pm 2.00\mathbf{i}$ |
| $\pm 0.77\mathbf{i}$ | $\pm 1.85\mathbf{i}$ | $\mp 2.00\mathbf{i}$ |
| $\pm 1.08\mathbf{i}$ | $\mp 2.61\mathbf{i}$ | 0.00 |

- Initial conditions to excite just the three modes:

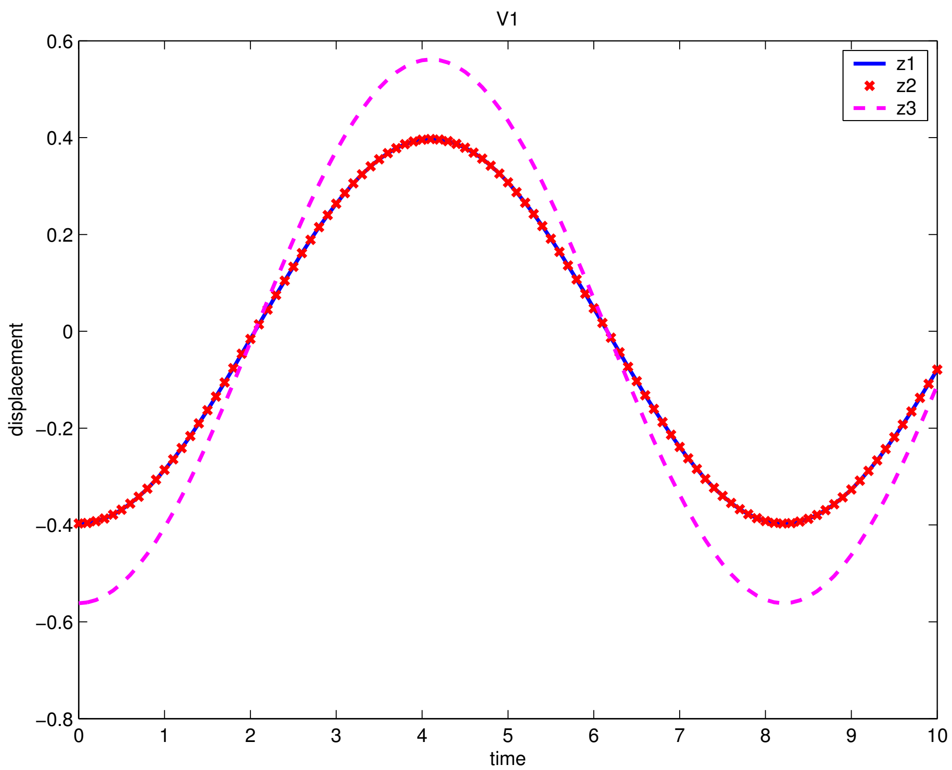
$$x_i(0) = \alpha_1 \text{Re}(v_i) + \alpha_2 \text{Im}(v_1) \quad \forall \alpha_j \in \mathbb{R}$$

  – Simulation using $\alpha_1 = 1$, $\alpha_2 = 0$

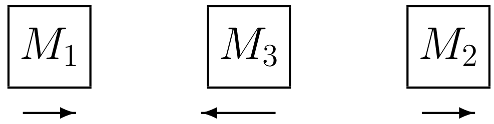
- Visualization important for correct physical interpretation
- Mode 1 $\lambda_1 = \pm 0.77\mathbf{i}$



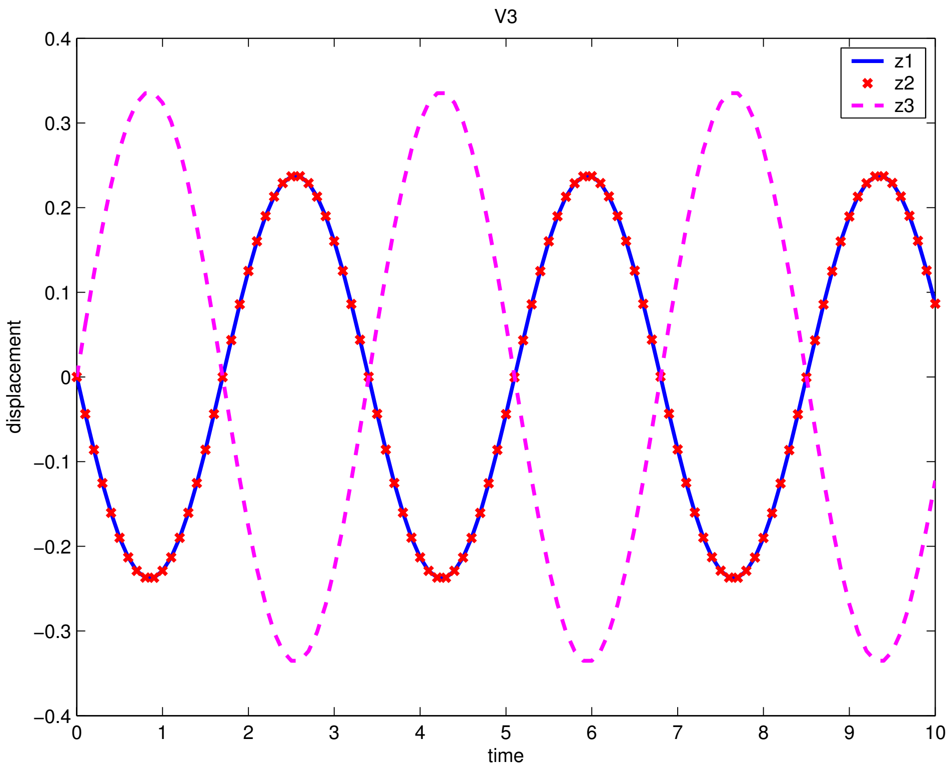

  – Lowest frequency mode, all masses move in same direction
  – Middle mass has higher amplitude motions $z_3$, motions all in phase



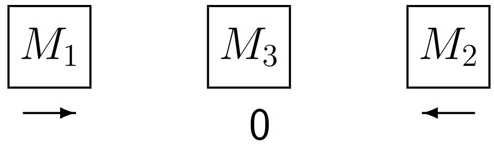
- Mode 2 $\lambda_2 = \pm 1.85\mathbf{i}$



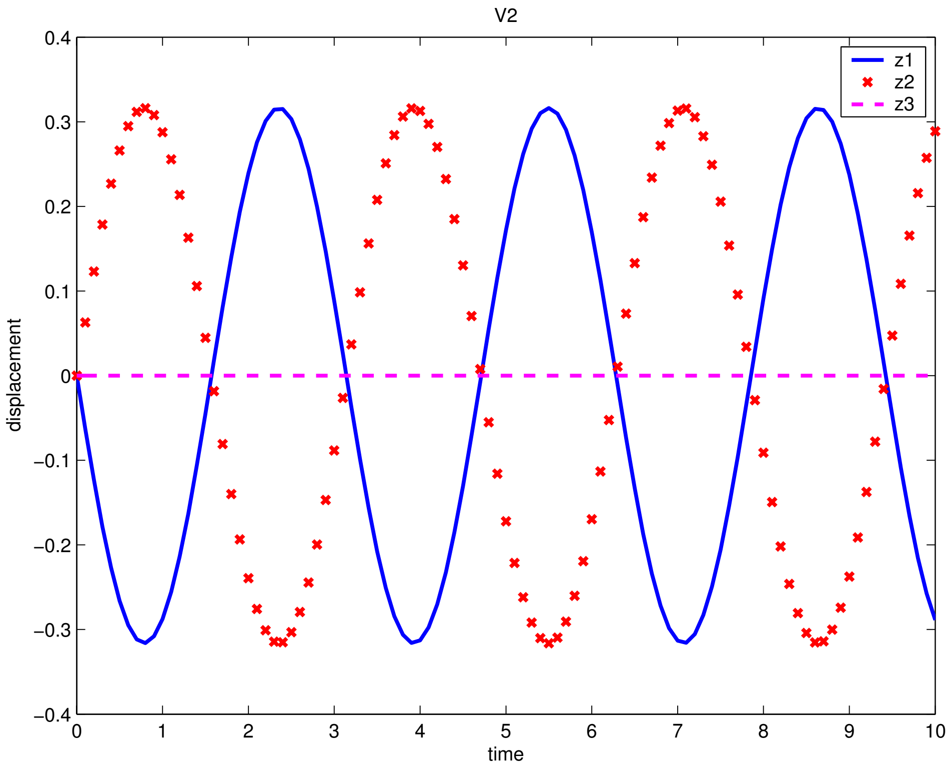

  - Middle frequency mode has middle mass moving in opposition to two end masses
  - Again middle mass has higher amplitude motions $z_3$



- Mode 3 $\lambda_3 = \pm 2.00\mathbf{i}$




  – Highest frequency mode, has middle mass stationary, and other two masses in opposition




- Eigenvectors with that correspond with more constrained motion of the system are associated with higher frequency eigenvalues