16.333 Lecture 4

Aircraft Dynamics

- Aircraft nonlinear EOM
- Linearization – dynamics
- Linearization – forces & moments
- Stability derivatives and coefficients

Aircraft Dynamics

- Note can develop good approximation of key aircraft motion (Phugoid) using simple balance between kinetic and potential energies.
- Consider an aircraft in steady, level flight with speed U_0 and height h_0. The motion is perturbed slightly so that

$$U_0 \rightarrow U = U_0 + u \quad (1)$$

$$h_0 \rightarrow h = h_0 + \Delta h \quad (2)$$

- Assume that $E = \frac{1}{2}mU^2 + mgh$ is constant before and after the perturbation. It then follows that $u \approx -\frac{g\Delta h}{U_0}$
- From Newton's laws we know that, in the vertical direction

$$m\ddot{h} = L - W$$

where weight $W = mg$ and lift $L = \frac{1}{2}\rho S C_L U^2$ (S is the wing area). We can then derive the equations of motion of the aircraft:

$$m\ddot{h} = L - W = \frac{1}{2}\rho S C_L (U^2 - U_0^2) \quad (3)$$

$$= \frac{1}{2}\rho S C_L ((U_0 + u)^2 - U_0^2) \approx \frac{1}{2}\rho S C_L (2uU_0) \quad (4)$$

$$\approx -\rho S C_L \left(\frac{g\Delta h}{U_0} U_0\right) = -(\rho S C_L g)\Delta h \quad (5)$$

Since $\ddot{h} = \Delta\ddot{h}$ and for the original equilibrium flight condition $L = W = \frac{1}{2}(\rho S C_L)U_0^2 = mg$, we get that

$$\frac{\rho S C_L g}{m} = 2\left(\frac{g}{U_0}\right)^2$$

Combine these result to obtain:

$$\Delta\ddot{h} + \Omega^2 \Delta h = 0 \quad , \quad \Omega \approx \frac{g}{U_0}\sqrt{2}$$

- These equations describe an oscillation (called the phugoid oscillation) of the altitude of the aircraft about it nominal value.
  - Only approximate natural frequency (Lanchester), but value close.

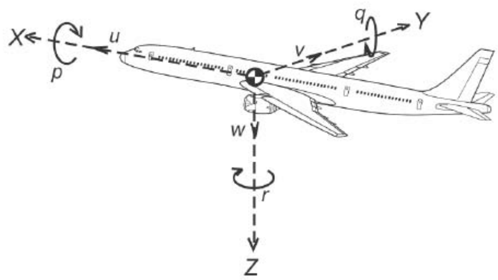
- The basic dynamics are:

$$\vec{F} = m\dot{\vec{v}}_c^{\,I} \quad \text{and} \quad \vec{T} = \dot{\vec{H}}^{\,I}$$

$$\Rightarrow \ \frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + \ {}^{BI}\vec{\omega} \times \vec{v}_c \qquad \textbf{Transport Thm.}$$

$$\Rightarrow \ \vec{T} = \dot{\vec{H}}^{\,B} + \ {}^{BI}\vec{\omega} \times \vec{H}$$

- Basic assumptions are:



1. Earth is an inertial reference frame
2. A/C is a rigid body
3. Body frame **B** fixed to the aircraft $(\vec{i}, \vec{j},$

- Instantaneous mapping of $\vec{v}_c$ and ${}^{BI}\vec{\omega}$ into the body frame:

$${}^{BI}\vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k} \qquad \vec{v}_c = U\vec{i} + V\vec{j} + W\vec{k}$$

$$\Rightarrow \ {}^{BI}\omega_B = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad \Rightarrow (v_c)_B = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

- By symmetry, we can show that $I_{xy} = I_{yz} = 0$, but value of I_{xz} depends on specific frame selected. Instantaneous mapping of the angular momentum

$$\vec{H} = H_x\vec{i} + H_y\vec{j} + H_z\vec{k}$$

into the Body Frame given by

$$H_B = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

- The overall equations of motion are then:

$$\frac{1}{m}\vec{F} = \dot{\vec{v}}_c^{\,B} + {}^{BI}\vec{\omega} \times \vec{v}_c$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix}$$

$$= \begin{bmatrix} \dot{U} + QW - RV \\ \dot{V} + RU - PW \\ \dot{W} + PV - QU \end{bmatrix}$$

$$\vec{T} = \dot{\vec{H}}^{B} + {}^{BI}\vec{\omega} \times \vec{H}$$

$$\Rightarrow \begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} \\ I_{yy}\dot{Q} \\ I_{zz}\dot{R} + I_{xz}\dot{P} \end{bmatrix} + \begin{bmatrix} 0 & -R & Q \\ R & 0 & -P \\ -Q & P & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

- Clearly these equations are very nonlinear and complicated, and we have not even said where $\vec{F}$ and $\vec{T}$ come from. $\Longrightarrow$ Need to linearize!!
  - Assume that the aircraft is flying in an *equilibrium condition* and we will linearize the equations about this nominal flight condition.

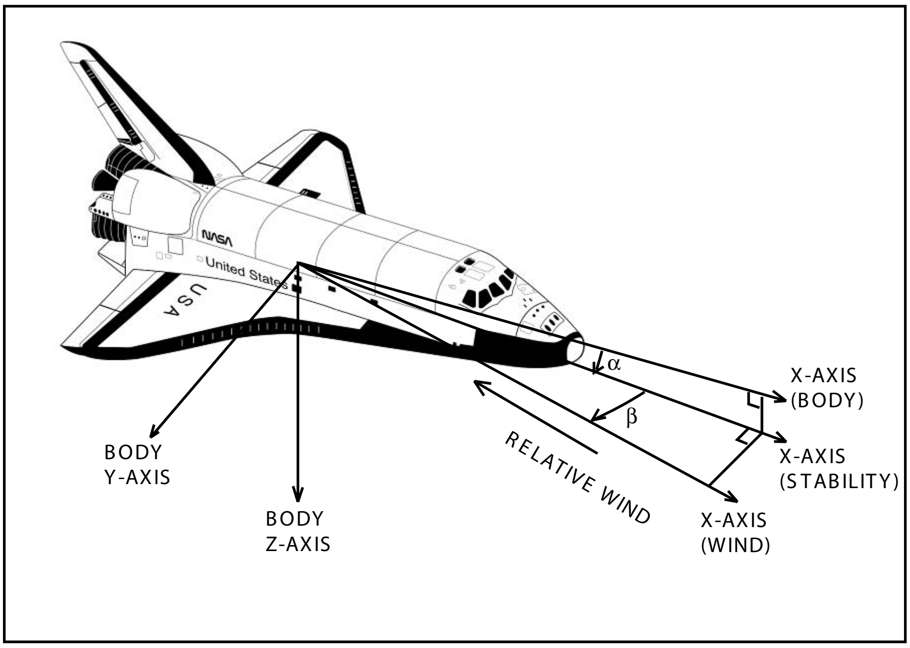
Axes

- But first we need to be a little more specific about which *Body Frame* we are going use. Several standards:

  1. **Body Axes** - X aligned with fuselage nose. Z perpendicular to X in plane of symmetry (down). Y perpendicular to XZ plane, to the right.

  2. **Wind Axes** - X aligned with $\vec{v}_c$. Z perpendicular to X (pointed down). Y perpendicular to XZ plane, off to the right.

  3. **Stability Axes** - X aligned with projection of $\vec{v}_c$ into the fuselage plane of symmetry. Z perpendicular to X (pointed down). Y same.



- Advantages to each, but typically use the **stability axes**.

  – In different *flight equilibrium conditions*, the axes will be oriented differently with respect to the A/C principal axes ⇒ need to transform (rotate) the principal inertia components between the frames.

  – When vehicle undergoes motion with respect to the equilibrium, **Stability Axes remain fixed to airplane as if painted on.**

- Can linearize about various steady state conditions of flight.
  - For steady state flight conditions must have
  $$\vec{F} = \vec{F}_{\text{aero}} + \vec{F}_{\text{gravity}} + \vec{F}_{\text{thrust}} = 0 \ \text{ and } \vec{T} = 0$$
    - ◇ So for equilibrium condition, forces balance on the aircraft $L = W$ and $T = D$
  - Also assume that $\dot{P} = \dot{Q} = \dot{R} = \dot{U} = \dot{V} = \dot{W} = 0$
  - Impose additional constraints that depend on **flight condition:**
    - ◇ Steady wings-level flight $\rightarrow \Phi = \dot{\Phi} = \dot{\Theta} = \dot{\Psi} = 0$

- **Key Point:** While nominal forces and moments balance to zero, motion about the equilibrium condition results in perturbations to the forces/moments.
  - Recall from basic flight dynamics that lift $L_0^f = C_{L_\alpha} \alpha_0$ where:
    - ◇ C_{L_α} = *lift curve slope* – function of the equilibrium condition
    - ◇ α_0 = nominal *angle of attack* (angle that wing meets air flow)
  - But, as the vehicle moves about the equilibrium condition, would expect that the angle of attack will change
  $$\alpha = \alpha_0 + \Delta\alpha$$
  - Thus the lift forces will also be perturbed
  $$L^f = C_{L_\alpha}(\alpha_0 + \Delta\alpha) = L_0^f + \Delta L^f$$

- Can extend this idea to all dynamic variables and how they influence all aerodynamic forces and moments

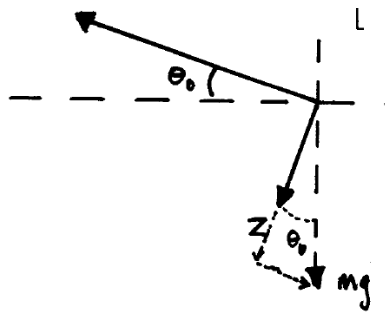
Gravity Forces

- Gravity acts through the CoM in vertical direction (inertial frame +Z)
  - Assume that we have a non-zero pitch angle Θ_0
  - Need to map this force into the body frame
  - Use the Euler angle transformation (2–15)

$$F_B^g = T_1(\Phi)T_2(\Theta)T_3(\Psi) \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} = mg \begin{bmatrix} -\sin\Theta \\ \sin\Phi\cos\Theta \\ \cos\Phi\cos\Theta \end{bmatrix}$$

- For symmetric steady state flight equilibrium, we will typically assume that $\Theta \equiv \Theta_0$, $\Phi \equiv \Phi_0 = 0$, so

$$F_B^g = mg \begin{bmatrix} -\sin\Theta_0 \\ 0 \\ \cos\Theta_0 \end{bmatrix}$$



- Use Euler angles to specify vehicle rotations with respect to the Earth frame

$$\begin{aligned} \dot{\Theta} &= Q\cos\Phi - R\sin\Phi \\ \dot{\Phi} &= P + Q\sin\Phi\tan\Theta + R\cos\Phi\tan\Theta \\ \dot{\Psi} &= (Q\sin\Phi + R\cos\Phi)\sec\Theta \end{aligned}$$

  - Note that if $\Phi \approx 0$, then $\dot{\Theta} \approx Q$

- **Recall:** $\Phi \approx$ Roll, $\Theta \approx$ Pitch, and $\Psi \approx$ Heading.

Linearization

- Define the **trim** angular rates and velocities

$$^{BI}\omega_B^o = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \qquad (v_c)_B^o = \begin{bmatrix} U_o \\ 0 \\ 0 \end{bmatrix}$$

which are associated with the flight condition. In fact, these define the type of equilibrium motion that we linearize about. **Note:**

  – $W_0 = 0$ since we are using the stability axes, and
  – $V_0 = 0$ because we are assuming symmetric flight

- Proceed with linearization of the dynamics for various flight conditions

	Nominal Velocity	Perturbed Velocity	⇒	Perturbed Acceleration
Velocities	U_0,	$U = U_0 + u$	⇒	$\dot{U} = \dot{u}$
	$W_0 = 0$,	$W = w$	⇒	$\dot{W} = \dot{w}$
	$V_0 = 0$,	$V = v$	⇒	$\dot{V} = \dot{v}$
Angular Rates	$P_0 = 0$,	$P = p$	⇒	$\dot{P} = \dot{p}$
	$Q_0 = 0$,	$Q = q$	⇒	$\dot{Q} = \dot{q}$
	$R_0 = 0$,	$R = r$	⇒	$\dot{R} = \dot{r}$
Angles	Θ_0,	$\Theta = \Theta_0 + \theta$	⇒	$\dot{\Theta} = \dot{\theta}$
	$\Phi_0 = 0$,	$\Phi = \phi$	⇒	$\dot{\Phi} = \dot{\phi}$
	$\Psi_0 = 0$,	$\Psi = \psi$	⇒	$\dot{\Psi} = \dot{\psi}$

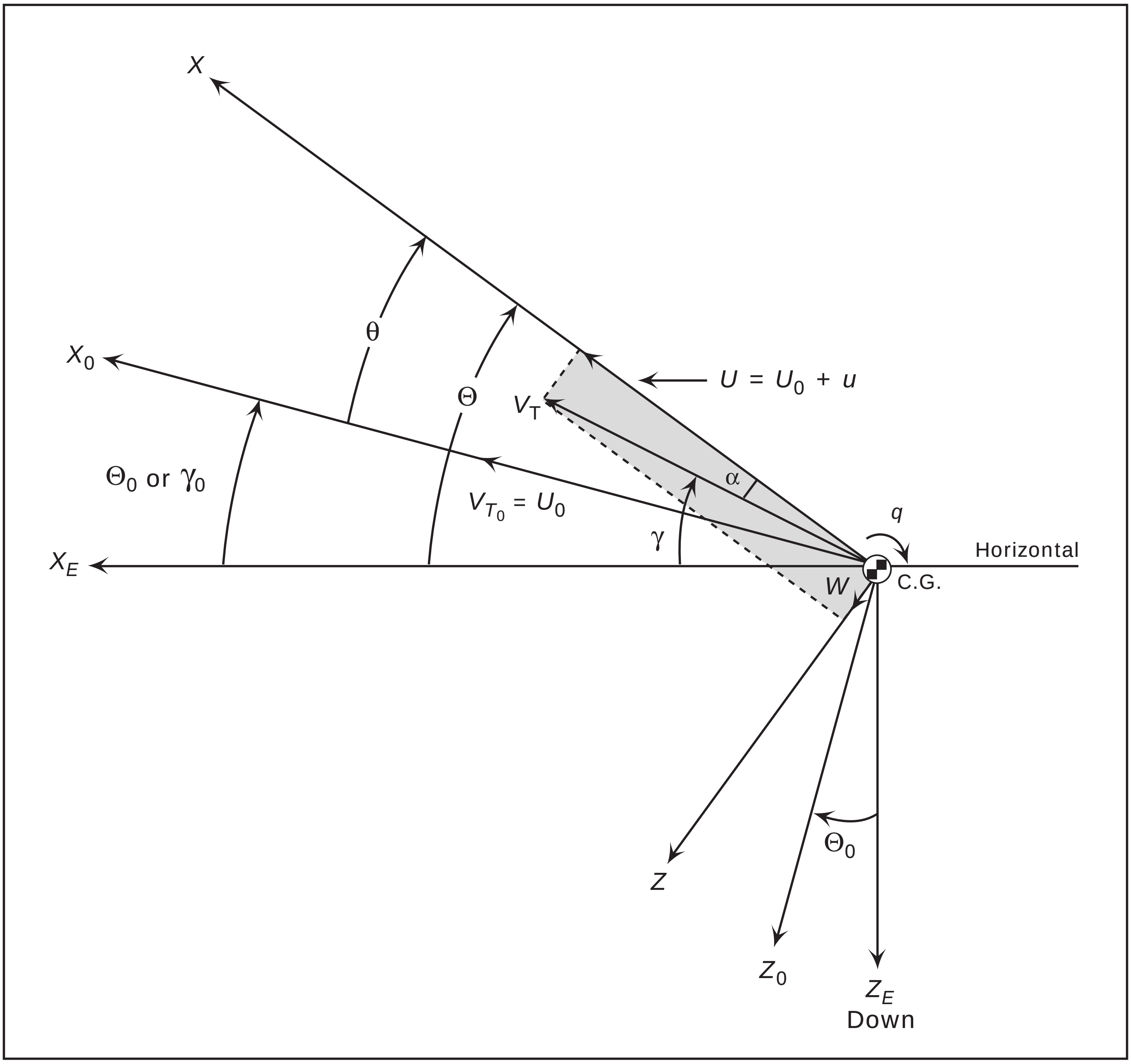


Figure 1: Perturbed Axes. The equilibrium condition was that the aircraft was angled up by Θ_0 with velocity $V_{T0} = U_0$. The vehicle's motion has been perturbed $(X_0 \to X)$ so that now $\Theta = \Theta_0 + \theta$ and the velocity is $V_T \neq V_{T0}$. Note that V_T is no longer aligned with the X-axis, resulting in a non-zero u and w. The angle γ is called the **flight path angle,** and it provides a measure of the angle of the velocity vector to the inertial horizontal axis.

- **Linearization for symmetric flight**
  $U = U_0 + u,\ V_0 = W_0 = 0,\ P_0 = Q_0 = R_0 = 0.$

  Note that the forces and moments are also perturbed.

$$\frac{1}{m}[X_0 + \Delta X] = \dot{U} + QW - RV \approx \dot{u} + qw - rv \approx \dot{u}$$

$$\frac{1}{m}[Y_0 + \Delta Y] = \dot{V} + RU - PW$$
$$\approx \dot{v} + r(U_0 + u) - pw \approx \dot{v} + rU_0$$

$$\frac{1}{m}[Z_0 + \Delta Z] = \dot{W} + PV - QU \approx \dot{w} + pv - q(U_0 + u)$$
$$\approx \dot{w} - qU_0$$

$$\Rightarrow \frac{1}{m}\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} \dot{u} \\ \dot{v} + rU_0 \\ \dot{w} - qU_0 \end{bmatrix} \quad \begin{matrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{matrix}$$

- Attitude motion:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{P} + I_{xz}\dot{R} & +QR(I_{zz} - I_{yy}) + PQI_{xz} \\ I_{yy}\dot{Q} & +PR(I_{xx} - I_{zz}) + (R^2 - P^2)I_{xz} \\ I_{zz}\dot{R} + I_{xz}\dot{P} & +PQ(I_{yy} - I_{xx}) - QRI_{xz} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Delta L \\ \Delta M \\ \Delta N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{yy}\dot{q} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix} \quad \begin{matrix} \mathbf{4} \\ \mathbf{5} \\ \mathbf{6} \end{matrix}$$

- Key aerodynamic parameters are also perturbed:

$$\begin{aligned}
&\text{Total Velocity} \\
V_T &= ((U_0+u)^2+v^2+w^2)^{1/2} \approx U_0+u \\
&\text{Perturbed Sideslip angle} \\
\beta &= \sin^{-1}(v/V_T) \approx v/U_0 \\
&\text{Perturbed Angle of Attack} \\
\alpha_x &= \tan^{-1}(w/U) \approx w/U_0
\end{aligned}$$

- To understand these equations in detail, and the resulting impact on the vehicle dynamics, we must investigate the terms $\Delta X \ldots \Delta N$.
  - We must also address the left-hand side ($\vec{F}$, $\vec{T}$)
  - **Net** forces and moments must be zero in equilibrium condition.
  - Aerodynamic and Gravity forces are a function of equilibrium condition **AND** the perturbations about this equilibrium.

- Predict the changes to the aerodynamic forces and moments using a first order expansion in the key flight parameters

$$\begin{aligned}
\Delta X &= \frac{\partial X}{\partial U}\mathbf{\Delta U} + \frac{\partial X}{\partial W}\mathbf{\Delta W} + \frac{\partial X}{\partial \dot{W}}\mathbf{\Delta \dot{W}} + \frac{\partial X}{\partial \Theta}\Delta\Theta + \ldots + \frac{\partial X^g}{\partial \Theta}\Delta\Theta + \Delta X^c \\
&= \frac{\partial X}{\partial U}\mathbf{u} + \frac{\partial X}{\partial W}\mathbf{w} + \frac{\partial X}{\partial \dot{W}}\mathbf{\dot{w}} + \frac{\partial X}{\partial \Theta}\theta + \ldots + \frac{\partial X^g}{\partial \Theta}\theta + \Delta X^c
\end{aligned}$$

- $\frac{\partial X}{\partial U}$ called **stability derivative** – evaluated at eq. condition.

- Gives dimensional form; non-dimensional form available in tables.

- Clearly approximation since ignores lags in the aerodynamics forces (assumes that forces only function of instantaneous values)

Stability Derivatives

- First proposed by Bryan (1911) – has proven to be a **very** effective way to analyze the aircraft flight mechanics – well supported by numerous flight test comparisons.
- The forces and torques acting on the aircraft are very complex nonlinear functions of the flight equilibrium condition and the perturbations from equilibrium.
  - Linearized expansion can involve many terms $u, \dot{u}, \ddot{u}, \ldots, w, \dot{w}, \ddot{w}, \ldots$
  - Typically only retain a few terms to capture the dominant effects.
- Dominant behavior most easily discussed in terms of the:
  - Symmetric variables: U, W, Q & forces/torques: X, Z, and M
  - Asymmetric variables: V, P, R & forces/torques: Y, L, and N
- Observation – for truly symmetric flight Y, L, and N will be exactly **zero** for any value of U, W, Q

  $\Rightarrow$ Derivatives of asymmetric forces/torques with respect to the symmetric motion variables are **zero**.
- Further (convenient) assumptions:
  1. Derivatives of symmetric forces/torques with respect to the asymmetric motion variables are small and can be neglected.
  2. We can neglect derivatives with respect to the derivatives of the motion variables, but keep $\partial Z/\partial \dot{w}$ and $M_{\dot{w}} \equiv \partial M/\partial \dot{w}$ (aerodynamic lag involved in forming new pressure distribution on the wing in response to the perturbed angle of attack)
  3. $\partial X/\partial q$ is negligibly small.

$\partial()/\partial()$	**X**	**Y**	**Z**	**L**	**M**	**N**
u	•	0	•	0	•	0
v	0	•	0	•	0	•
w	•	0	•	0	•	0
p	0	•	0	•	0	•
q	≈ 0	0	•	0	•	0
r	0	•	0	•	0	•

- Note that we must also find the perturbation gravity and thrust forces and moments

$$\left.\frac{\partial X^g}{\partial \Theta}\right|_0 = -mg\cos\Theta_0 \qquad \left.\frac{\partial Z^g}{\partial \Theta}\right|_0 = -mg\sin\Theta_0$$

- **Aerodynamic summary:**

1A $\Delta X = \left(\frac{\partial X}{\partial U}\right)_0 u + \left(\frac{\partial X}{\partial W}\right)_0 w \Rightarrow \Delta X \sim u,\ \alpha_x \approx w/U_0$

2A $\Delta Y \sim \beta \approx v/U_0,\ p,\ r$

3A $\Delta Z \sim u,\ \alpha_x \approx w/U_0,\ \dot{\alpha}_x \approx \dot{w}/U_0,\ q$

4A $\Delta L \sim \beta \approx v/U_0,\ p,\ r$

5A $\Delta M \sim u,\ \alpha_x \approx w/U_0,\ \dot{\alpha}_x \approx \dot{w}/U_0,\ q$

6A $\Delta N \sim \beta \approx v/U_0,\ p,\ r$

- Result is that, with these force, torque approximations, equations **1**, **3**, **5** decouple from **2**, **4**, **6**

  – **1**, **3**, **5** are the **longitudinal dynamics** in u, w, and q

$$\begin{bmatrix} \Delta X \\ \Delta Z \\ \Delta M \end{bmatrix} = \begin{bmatrix} m\dot{u} \\ m(\dot{w} - qU_0) \\ I_{yy}\dot{q} \end{bmatrix}$$

$$\approx \begin{bmatrix} \left(\frac{\partial X}{\partial U}\right)_0 u + \left(\frac{\partial X}{\partial W}\right)_0 w + \left(\frac{\partial X^g}{\partial \Theta}\right)_0 \theta + \Delta X^c \\ \left(\frac{\partial Z}{\partial U}\right)_0 u + \left(\frac{\partial Z}{\partial W}\right)_0 w + \left(\frac{\partial Z}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial Z}{\partial Q}\right)_0 q + \left(\frac{\partial Z^g}{\partial \Theta}\right)_0 \theta + \Delta Z^c \\ \left(\frac{\partial M}{\partial U}\right)_0 u + \left(\frac{\partial M}{\partial W}\right)_0 w + \left(\frac{\partial M}{\partial \dot{W}}\right)_0 \dot{w} + \left(\frac{\partial M}{\partial Q}\right)_0 q + \Delta M^c \end{bmatrix}$$

  – **2**, **4**, **6** are the **lateral dynamics** in v, p, and r

$$\begin{bmatrix} \Delta Y \\ \Delta L \\ \Delta N \end{bmatrix} = \begin{bmatrix} m(\dot{v} + rU_0) \\ I_{xx}\dot{p} + I_{xz}\dot{r} \\ I_{zz}\dot{r} + I_{xz}\dot{p} \end{bmatrix}$$

$$\approx \begin{bmatrix} \left(\frac{\partial Y}{\partial V}\right)_0 v + \left(\frac{\partial Y}{\partial P}\right)_0 p + \left(\frac{\partial Y}{\partial R}\right)_0 r + +\Delta Y^c \\ \left(\frac{\partial L}{\partial V}\right)_0 v + \left(\frac{\partial L}{\partial P}\right)_0 p + \left(\frac{\partial L}{\partial R}\right)_0 r + \Delta L^c \\ \left(\frac{\partial N}{\partial V}\right)_0 v + \left(\frac{\partial N}{\partial P}\right)_0 p + \left(\frac{\partial N}{\partial R}\right)_0 r + \Delta N^c \end{bmatrix}$$

Basic Stability Derivative Derivation

- Consider changes in the drag force with forward speed U

$$\begin{aligned} D &= 1/2\rho V_T^2 S C_D \\ V_T^2 &= (u_0+u)^2 + v^2 + w^2 \\ \frac{\partial V_T^2}{\partial u} &= 2(u_0+u) \Rightarrow \left(\frac{\partial V_T^2}{\partial u}\right)_0 = 2u_0 \end{aligned}$$

Note: $\left(\frac{\partial V_T^2}{\partial v}\right)_0 = 0$ and $\left(\frac{\partial V_T^2}{\partial w}\right)_0 = 0$

At reference condition:

$$\begin{aligned} \Rightarrow D_u \equiv \left(\frac{\partial D}{\partial u}\right)_0 &= \left\{\frac{\partial}{\partial u}\left(\frac{\rho V_T^2 S C_D}{2}\right)\right\}_0 \\ &= \frac{\rho S}{2}\left(u_0^2\left(\frac{\partial C_D}{\partial u}\right)_0 + C_{D_0}\left(\frac{\partial V_T^2}{\partial u}\right)_0\right) \\ &= \frac{\rho S}{2}\left(u_0^2\left(\frac{\partial C_D}{\partial u}\right)_0 + 2u_0 C_{D_0}\right) \end{aligned}$$

– Note $\frac{\partial D}{\partial u}$ is the **stability derivative**, which is **dimensional**.

- Define nondimensional **stability coefficient** C_{D_u} as derivative of C_D with respect to a **nondimensional velocity** u/u_0

$$C_D = \frac{D}{\frac{1}{2}\rho V_T^2 S} \quad \Rightarrow \quad C_{D_u} \equiv \left(\frac{\partial C_D}{\partial u/u_0}\right)_0 \quad \text{and} \quad \mathbf{C_{D_0} \equiv (C_D)_0}$$

– So $(\bullet)_0$ corresponds to the variable at its equilibrium condition.

- Nondimensionalize:

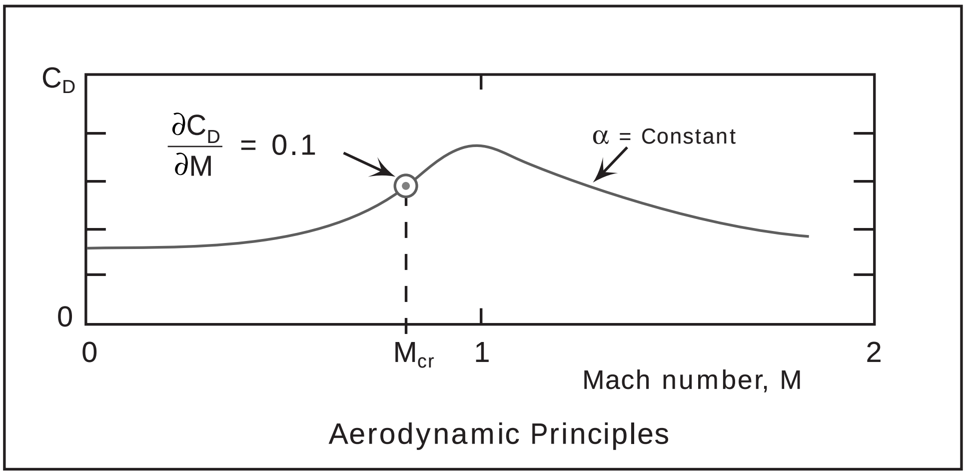
$$\begin{aligned}\left(\frac{\partial D}{\partial u}\right)_0 &= \frac{\rho S u_0}{2}\left(u_0\left(\frac{\partial C_D}{\partial u}\right)_0 + 2C_{D_0}\right)\\ &= \frac{QS}{u_0}\left(\left(\frac{\partial C_D}{\partial u/u_0}\right)_0 + 2C_{D_0}\right)\\ \left(\frac{u_0}{QS}\right)\left(\frac{\partial D}{\partial u}\right)_0 &= (C_{D_u} + 2C_{D_0})\end{aligned}$$

  So given stability coefficient, can compute the drag force increment.

- Note that Mach number has a significant effect on the drag:

$$C_{D_u} = \left(\frac{\partial C_D}{\partial u/u_0}\right)_0 = \left(\frac{u_0}{a}\frac{\partial C_D}{\partial\left(\frac{u}{a}\right)}\right)_0 = \mathbf{M}\frac{\partial C_D}{\partial \mathbf{M}}$$

  where $\frac{\partial C_D}{\partial \mathbf{M}}$ can be estimated from empirical results/tables.



- Thrust forces

$$C_{T_u} = \left(\frac{\partial C_T}{\partial u/u_0}\right)_0 \quad \Rightarrow \quad \left(\frac{\partial T}{\partial u}\right)_0 = C_{T_u}\frac{1}{u_0}QS$$

  – For a glider, $C_{T_u} = 0$

  – For a jet, $C_{T_u} \approx 0$

  – For a prop plane, $C_{T_u} = -C_{D_0}$

- Lift forces similar to drag

$$L = 1/2 \rho V_T^2 S C_L$$

$$\Rightarrow \left(\frac{\partial L}{\partial u}\right)_0 = \frac{\rho S u_0}{2}\left(u_0 \left(\frac{\partial C_L}{\partial u}\right)_0 + 2C_{L_0}\right)$$
$$= \frac{QS}{u_0}\left(\left(\frac{\partial C_L}{\partial u/u_0}\right)_0 + 2C_{L_0}\right)$$

$$\left(\frac{u_0}{QS}\right)\left(\frac{\partial L}{\partial u}\right)_0 = (C_{L_u} + 2C_{L_0})$$

where C_{L_0} is the lift coefficient for the eq. condition and $C_{L_u} = \mathbf{M}\frac{\partial C_L}{\partial \mathbf{M}}$ as before. From aerodynamic theory, we have that

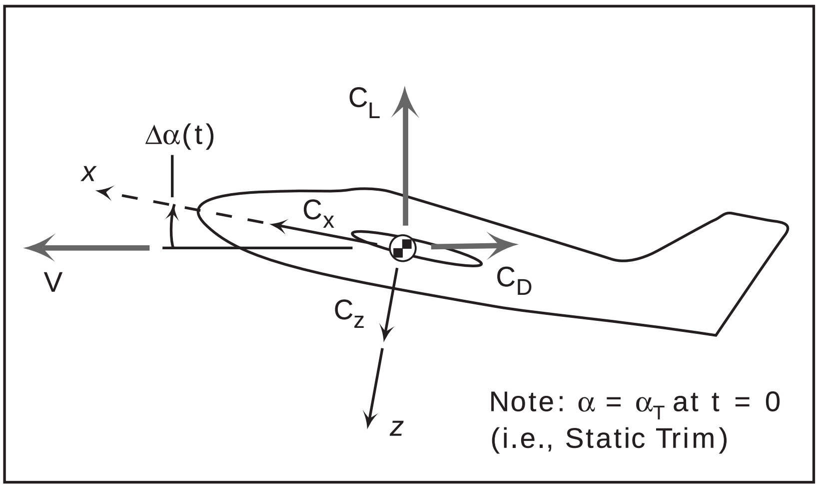
$$C_L = \frac{C_L|_{\mathbf{M}=0}}{\sqrt{1-\mathbf{M}^2}} \quad \Rightarrow \quad \frac{\partial C_L}{\partial \mathbf{M}} = \frac{\mathbf{M}}{1-\mathbf{M}^2} C_L$$

$$\Rightarrow C_{L_u} = \frac{\mathbf{M}^2}{1-\mathbf{M}^2} C_{L_0}$$

- α Derivatives: Now consider what happens with changes in the angle of attack. Take derivatives and evaluate at the reference condition:
  - Lift: $\Rightarrow C_{L_\alpha}$
  - Drag: $C_D = C_{D_{\min}} + \dfrac{C_L^2}{\pi e AR} \Rightarrow C_{D_\alpha} = \dfrac{2C_{L_0}}{\pi e AR} C_{L_\alpha}$

- Combine into X, Z Forces
  - At equilibrium, forces balance.
  - Use stability axes, so $\alpha_0 = 0$
  - Include the effect in the force balance of a change in α on the force rotations so that we can see the perturbations.
  - Assume perturbation α is small, so rotations are by $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha$

$$
\begin{aligned}
X &= T - D + L\alpha \\
Z &= -(L + D\alpha)
\end{aligned}
$$



- So, now consider the α derivatives of these forces:
$$\frac{\partial X}{\partial \alpha} = \frac{\partial T}{\partial \alpha} - \frac{\partial D}{\partial \alpha} + L + \alpha \frac{\partial L}{\partial \alpha}$$
  - Thrust variation with α very small $\left(\frac{\partial T}{\partial \alpha}\right)_0 \approx 0$
  - Apply at the reference condition ($\alpha = 0$), i.e. $C_{X_\alpha} = \left(\frac{\partial C_X}{\partial \alpha}\right)_0$
- Nondimensionalize and apply reference condition:

$$
\begin{aligned}
C_{X_\alpha} &= -C_{D_\alpha} + C_{L_0} \\
&= C_{L_0} - \frac{2C_{L_0}}{\pi e AR} C_{L_\alpha}
\end{aligned}
$$

- And for the Z direction
$$\frac{\partial Z}{\partial \alpha} = -D - \alpha\frac{\partial D}{\partial \alpha} - \frac{\partial L}{\partial \alpha}$$
Giving
$$C_{Z_\alpha} = -C_{D_0} - C_{L_\alpha}$$
- Recall that C_{M_α} was already found during the static analysis
- Can repeat this process for the other derivatives with respect to the forward speed.
- Forward speed:
$$\frac{\partial X}{\partial u} = \frac{\partial T}{\partial u} - \frac{\partial D}{\partial u} + \alpha\frac{\partial L}{\partial u}$$
So that
$$\left(\frac{u_0}{QS}\right)\left(\frac{\partial X}{\partial u}\right)_0 = \left(\frac{u_0}{QS}\right)\left(\frac{\partial T}{\partial u}\right)_0 - \left(\frac{u_0}{QS}\right)\left(\frac{\partial D}{\partial u}\right)_0$$
$$\Rightarrow C_{X_u} \equiv C_{T_u} - (C_{D_u} + 2C_{D_0})$$
- Similarly for the Z direction:
$$\frac{\partial Z}{\partial u} = -\frac{\partial L}{\partial u} - \alpha\frac{\partial D}{\partial u}$$
So that
$$\left(\frac{u_0}{QS}\right)\left(\frac{\partial Z}{\partial u}\right)_0 = -\left(\frac{u_0}{QS}\right)\left(\frac{\partial L}{\partial u}\right)_0$$
$$C_{Z_u} \equiv -(C_{L_u} + 2C_{L_0})$$
$$= -\frac{\mathbf{M^2}}{1-\mathbf{M^2}}C_{L_0} - 2C_{L_0}$$
- Many more derivatives to consider !

Summary

- Picked a specific Body Frame (stability axes) from the list of alternatives

  ⇒ Choice simplifies some of the linearization, but the inertias now change depending on the equilibrium flight condition.

- Since the nonlinear behavior is too difficult to analyze, we needed to consider the linearized dynamic behavior around a specific flight condition

  ⇒ Enables us to linearize RHS of equations of motion.

- Forces and moments also complicated nonlinear functions, so we linearized the LHS as well

  ⇒ Enables us to write the perturbations of the forces and moments in terms of the motion variables.

  – Engineering insight allows us to argue that many of the stability derivatives that couple the longitudinal (symmetric) and lateral (asymmetric) motions are small and can be ignored.

- Approach requires that you have the stability derivatives.
  – These can be measured or calculated from the aircraft plan form and basic aerodynamic data.