16.333: Lecture #3

Frame Rotations

Euler Angles

Quaternions

Euler Angles

- For general applications in 3D, often need to perform 3 separate rotations to relate our "inertial frame" to our "body frame"
  - Especially true for aircraft problems

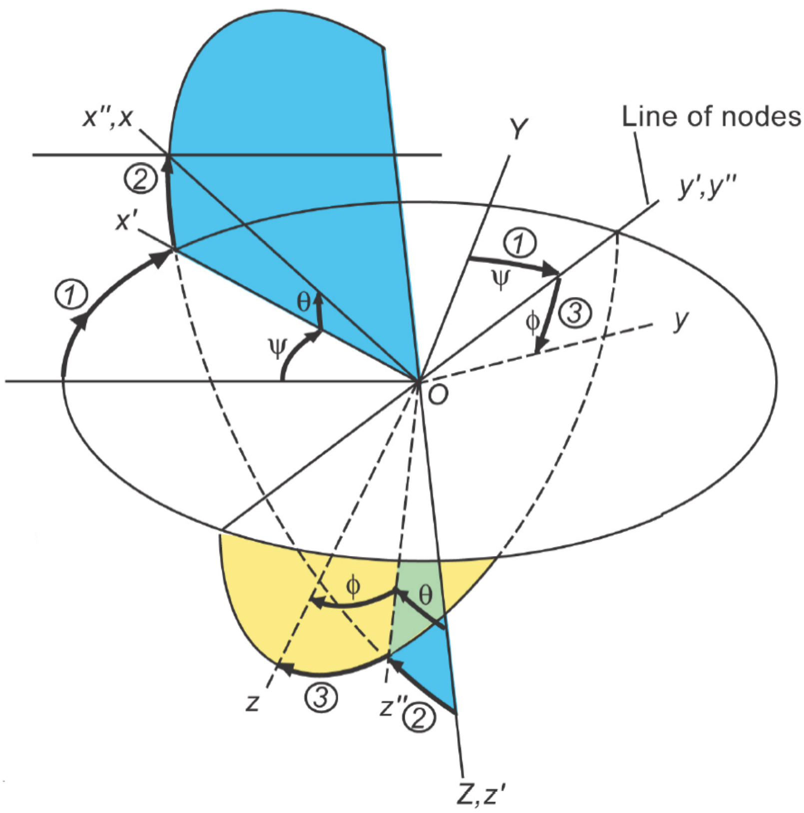
- There are many ways to do this set of rotations - with the variations be based on the order of the rotations
  - All would be acceptable
  - Some are more commonly used than others

- Standard: start with the body frame (x, y, z) aligned with the inertial (X, Y, Z), and then perform 3 rotations to re-orient the body frame.

① Rotate by ψ about $Z \Rightarrow x', y', z'$

② Rotate by θ about $y' \Rightarrow x'', y'', z''$

③ Rotate by ϕ about $x'' \Rightarrow x, y, z$



Euler angles:

- $\psi \sim$ Heading/yaw
- $\theta \sim$ Pitch
- $\phi \sim$ Roll

- Can write these rotations in a convenient form:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = T_3(\psi) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = T_2(\theta) \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = T_1(\phi) \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

which combines to give:

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= T_1(\phi) T_2(\theta) T_3(\psi) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \\ &= \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ -c\phi s\psi + s\phi s\theta c\psi & c\phi c\psi + s\phi s\theta s\psi & s\phi c\theta \\ s\phi s\psi + c\phi s\theta c\psi & -s\phi c\psi + c\phi s\theta s\psi & c\phi c\theta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \end{aligned}$$

- Note that the order that these rotations are applied matters and will greatly change the answer – matrix multiplies of T_i must be done consistently.

- To get the angular velocity in this case, we have to include three terms:

  ① $\dot{\psi}$ about Z

  ② $\dot{\theta}$ about y'

  ③ $\dot{\phi}$ about x''

  which we combine to get $\vec{\omega}$

- Want to write $\vec{\omega}$ in terms of its components in final frame (body)
  - Use the rotation matrices

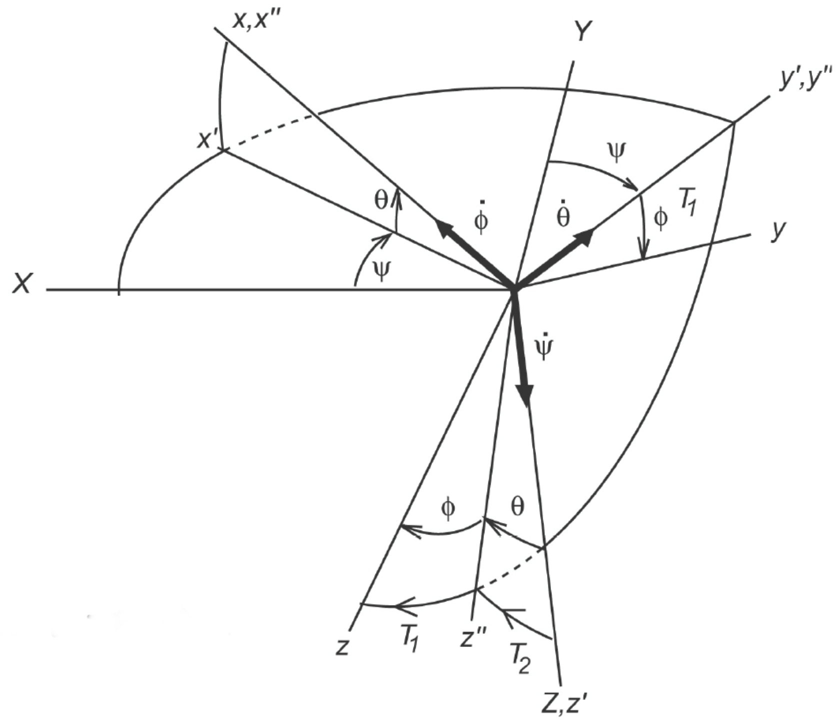
- Example: rotate $\dot{\psi}$ about $Z \equiv z'$
  - In terms of X, Y, Z, frame rotation rate has components $\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$, which is the same as in frame x', y', z'
  - To transform a vector from x', y', z' to x, y, z, need to use $T_1(\phi)T_2(\theta)$
  - Similar operation for $\dot{\theta}$ about $y' \equiv y''$ ⇒ use $T_1(\phi)$ on $\begin{bmatrix} 0 \\ \dot{\phi} \\ 0 \end{bmatrix}$

- Final result:

$$\omega_b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = T_1(\phi)T_2(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + T_1(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

- Visualization: Can write

$$\vec{\omega} = \vec{\dot{\phi}} + \vec{\dot{\theta}} + \vec{\dot{\psi}}$$



But $\vec{\dot{\phi}}, \vec{\dot{\theta}}, \vec{\dot{\psi}}$ do not form a mu
orthogonal triad

Need to form the orthogonal p
tions onto the body frame x,

$$\omega_b = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = T_1(\phi)T_2(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + T_1(\phi) \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

- Final form

$$\begin{aligned} \omega_x &= \dot{\phi} - \dot{\psi}\sin\theta \\ \omega_y &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ \omega_z &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi \end{aligned}$$

- With inverse:

$$\begin{aligned} \dot{\phi} &= \omega_x + [\omega_y\sin\phi + \omega_z\cos\phi]\tan\theta \\ \dot{\theta} &= \omega_y\cos\phi - \omega_z\sin\phi \\ \dot{\psi} &= [\omega_y\sin\phi + \omega_z\cos\phi]\sec\theta \end{aligned}$$

- Need to watch for singularities at $|\theta| = \pm 90^\circ$
- If we limit

$$\begin{aligned} 0 \leq &\ \psi \ \leq 2\pi \\ -\frac{\pi}{2} \leq &\ \theta \ \leq \frac{\pi}{2} \\ 0 \leq &\ \phi \ < 2\pi \end{aligned}$$

  then any possible orientation of the body can be obtained by performing the appropriate rotations in the order given.

- These are a pretty standard set of **Euler angles**

Quaternions

- Theorem by Euler states that any given sequence of rotations can be represented as a single rotation about a single fixed axis
- Quaterions provide a convenient parameterization of this effective axis and the rotation angle

$$\bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} \overline{E}\sin\zeta/2 \\ \cos\zeta/2 \end{bmatrix}$$

  where $\overline{E}$ is a unit vector and ζ is a positive rotation about $\overline{E}$

- Notes:
  - $\|\bar{b}\| = 1$ and thus there are only 3 degrees of freedom in this formulation as well
  - If $\bar{b}$ represents the rotational transformation from the reference frame a to reference frame b, the frame a is aligned with frame b when frame a is rotated by ζ radians about $\overline{E}$
- In terms of the Euler Angles:

$$\begin{aligned} \sin\theta &= -2(b_2b_4 + b_1b_3) \\ \phi &= \arctan 2\left[2(b_2b_3 - b_1b_4), 1 - 2(b_1^2 + b_2^2)\right] \\ \psi &= \arctan 2\left[2(b_1b_2 - b_3b_4), 1 - 2(b_2^2 + b_3^2)\right] \end{aligned}$$

- Pros:
  - Singularity free; Computationally efficient to do state propagation in time compared to Euler Angles
- Cons:
  - Far less intuitive - less appealing
- Refs: Kuipers, *Quaternions and rotation sequences*, 1999 Princeton University Press.