16.333: Lecture # 14

Equations of Motion in a Nonuniform Atmosphere

Gusts and Winds

Equations of Motion

- Analysis to date has assumed that the atmosphere is calm and fixed
  - Rarely true since we must contend with gusts and winds
  - Need to understand how these air motions impact our modeling of the aircraft.

- Must modify aircraft equations of motion since the aerodynamic forces and moments are functions of the **relative motion** between the aircraft and the atmosphere, and *not* of the **inertial** velocities.
  - Thus the LHS of the dynamics equations ($\vec{F} = m\vec{a}$) must be written in terms of the velocities relative to the atmosphere.
  - If u is the aircraft perturbation velocity (X direction), and u_g is the gust velocity in that direction, then the aircraft velocity with respect to the atmosphere is

$$u_a = u - u_g$$

- Now rewrite aerodynamic forces and moments in terms of aircraft velocity with respect to the atmosphere (see 4–11)

$$\begin{aligned}\Delta X &= \frac{\partial X}{\partial U}(u-u_g) + \frac{\partial X}{\partial W}(w-w_g) + \frac{\partial X}{\partial \dot{W}}(\dot{w}-\dot{w}_g) \\ &\quad + \frac{\partial X}{\partial Q}(q-q_g) + \ldots + \frac{\partial X}{\partial \Theta}\theta + \ldots + \frac{\partial X^g}{\partial \Theta}\theta + \Delta X^c\end{aligned}$$

  - The gravity terms $\frac{\partial X^g}{\partial \Theta}$ and control terms

$$\Delta X^c = X_{\delta_e}\delta_e + X_{\delta_p}\delta_p$$

    stay the same.

- The rotation gusts p_g, q_g, and r_g are caused by spatial variations in the gust components ⇒ rotary gusts are related to gradients of the vertical gust field

$$p_g = \frac{\partial w_g}{\partial y} \text{ and } q_g = \frac{\partial w_g}{\partial x}$$

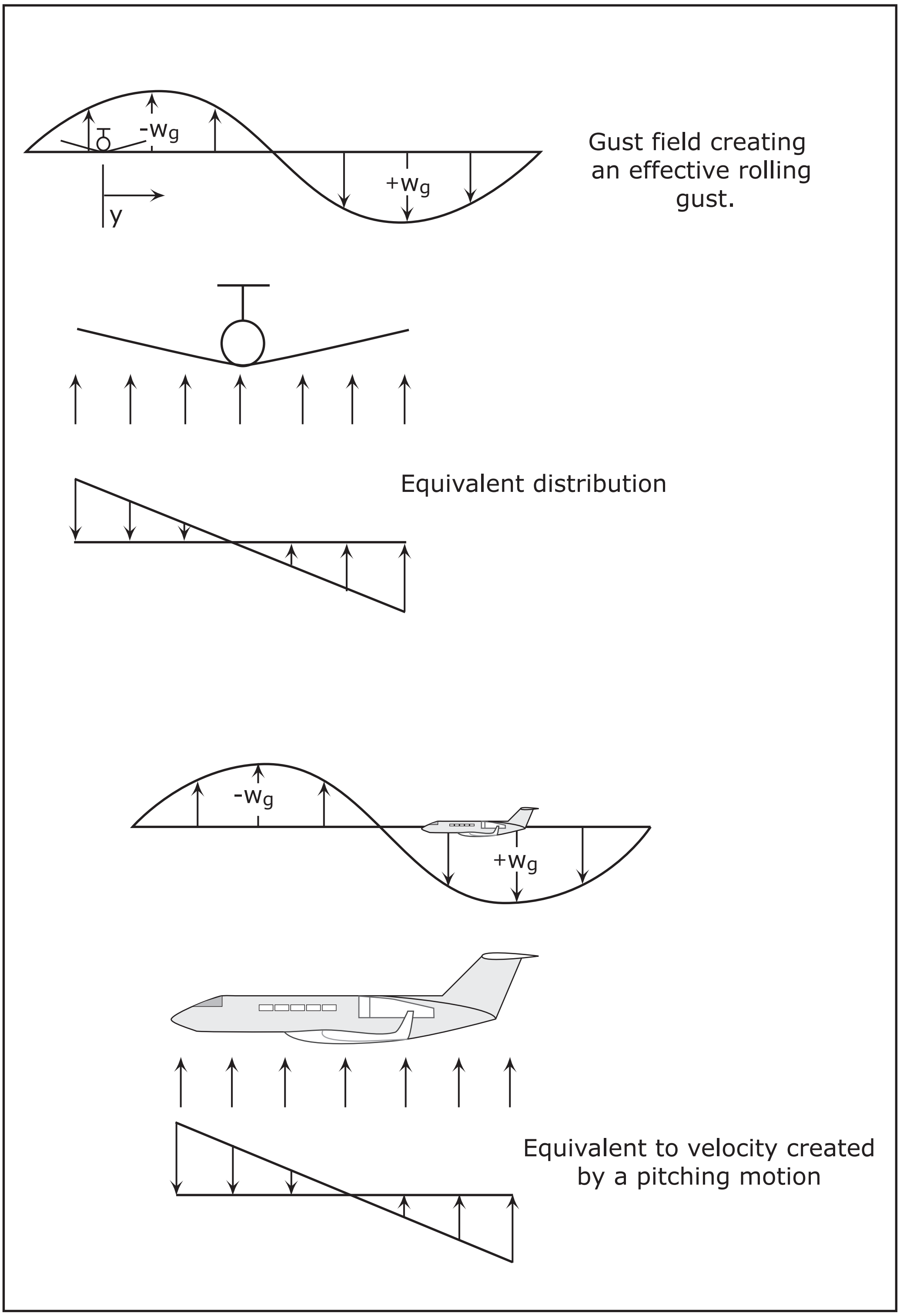

Figure 1: Gust Field creating an effective pitching gust.

- The next step is to include these new forces and moments in the equations of motion

$$\begin{bmatrix} m & 0 & 0 & 0 \\ 0 & m - Z_{\dot{w}} & 0 & 0 \\ 0 & -M_{\dot{w}} & I_{yy} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{w} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} X_u & X_w & 0 & -mg\cos\Theta_0 \\ Z_u & Z_w & Z_q + mU_0 & -mg\sin\Theta_0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

$$+ \begin{bmatrix} X_{\delta_e} & X_{\delta_p} \\ Z_{\delta_e} & Z_{\delta_p} \\ M_{\delta_e} & M_{\delta_p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}$$

$$+ \begin{bmatrix} -X_u & -X_w & 0 \\ -Z_u & -Z_w & 0 \\ -M_u & -M_w & -M_q \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_g \\ w_g \\ q_g \end{bmatrix}$$

$$\Rightarrow E\dot{x} = \hat{A}x + \hat{B}_u\mathbf{u} + \hat{B}_w\mathbf{w}$$

- Multiply through by E^{-1} to get new state space model

$$\dot{x} = Ax + B_u\mathbf{u} + B_w\mathbf{w}$$

  which has both control $\mathbf{u}$ and disturbance $\mathbf{w}$ inputs.

  – A similar operation can be performed for the lateral dynamics in terms of the disturbance inputs v_g, p_g, and r_g.

- Can now compute the response to specific types of gusts, such as a step or sinusoidal function, but usually are far more interested in the response to a *stochastic gust field*

Atmospheric Turbulence

- Can develop the best insight to how aircraft will behave with gust disturbances if we treat the disturbances as *random processes*.
  - What is a random process? Something (signal) that is random so that a deterministic description is not practical!!
  - But we can often describe the basic features of the process (e.g. mean value, how much it varies about the mean).

- Atmospheric turbulence is a random process, and the magnitude of the gust can only be described in terms of *statistical* properties.
  - For a random process $f(t)$, talk about the *mean square*

$$\overline{f^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f^2(t)\, dt$$

  as a measure of the *disturbance intensity* (how strong it is).

- Signal $f(t)$ can be decomposed into its Fourier components, so can use that to develop a frequency domain measure of disturbance strength
  - $\Phi(\omega) \approx$ that portion of $\overline{f^2(t)}$ that occurs in the frequency band

$$\omega \rightarrow \omega + d\omega$$

  - $\Phi(\omega)$ is called the *power spectral density*

- **Bottom line:** For a linear system $y = G(s)\mathbf{w}$, then

$$\Phi_y(\omega) = \Phi_w(\omega)|G(\mathbf{j}\omega)|^2$$

  $\Rightarrow$ Given an input disturbance spectral density (e.g. gusts), quite simple to predict expected output (e.g. ride comfort, wing loading).

Implementing a PSD in Matlab

- Simulink has a `Band-Limited White Noise` block that can be used for continuous systems.
  - Primary difference from the `Random Number` block is that this block produces output at a specific sample rate, which is related to the correlation time of the noise.

- Continuous white noise has a correlation time of 0 $\Rightarrow$ flat power spectral density (PSD), and a covariance of infinity.
  - Non-physical, but a useful approximation when the noise disturbance has a correlation time that is small relative to the natural bandwidth of the system.
  - Can simulate effect of white noise by using a random sequence with a correlation time much smaller than the shortest time constant of the system.

- `Band-Limited White Noise` block produces such a sequence where the *correlation time* of the noise is the *sample rate* of the block.
  - For accurate simulations, use a correlation time t_c much smaller than the fastest dynamics $f_{\max}$ of the system.

$$t_c \approx \frac{1}{100}\frac{2\pi}{f_{\max}}$$

- Power spectral densities for **Von Karman model** (see MIL-F-8785C)

$$\Phi_{u_g}(\Omega) = \sigma_u^2 \frac{2L_u}{\pi} \left[1 + (1.339 L_u \Omega)^2\right]^{-5/6}$$

where σ_u is the intensity measure of the disturbance, Ω is the spatial frequency variable, and L_u is a length scale of the disturbance.

– All scale parameters depend on day, altitude, and turbulence type.

– Different models available for each direction.

– Scale length parameter L (in feet) varies with altitude – **MIL-F-8785C** model valid up to 1000 feet

$$L_u = \frac{h}{(0.177 + 0.000823h)^{1.2}}$$

– Turbulence intensity for low altitude flight **MIL-F-8785C** as

$$\sigma_u = \frac{0.1 W_{20}}{(0.177 + 0.000823h)^{0.4}}$$

◇ where W_{20} is the wind speed as measured at 20 ft

◇ $W_{20} < 15$ knots is classified as "light" turbulence
$W_{20} \approx 30$ knots is "moderate"
$W_{20} > 45$ knots is "heavy"

– Standard approach: assume turbulence field fixed (*frozen*) in space
  ⇒ aircraft is just flying into it.
  ◇ Then can relate turbulence spatial frequency Ω to the temporal frequency ω that the aircraft would feel

$$\Omega \equiv \frac{\omega}{U_0} \quad \Rightarrow \quad \omega = \Omega U_0$$

– Used to model how the aircraft will behave in bumpy air.

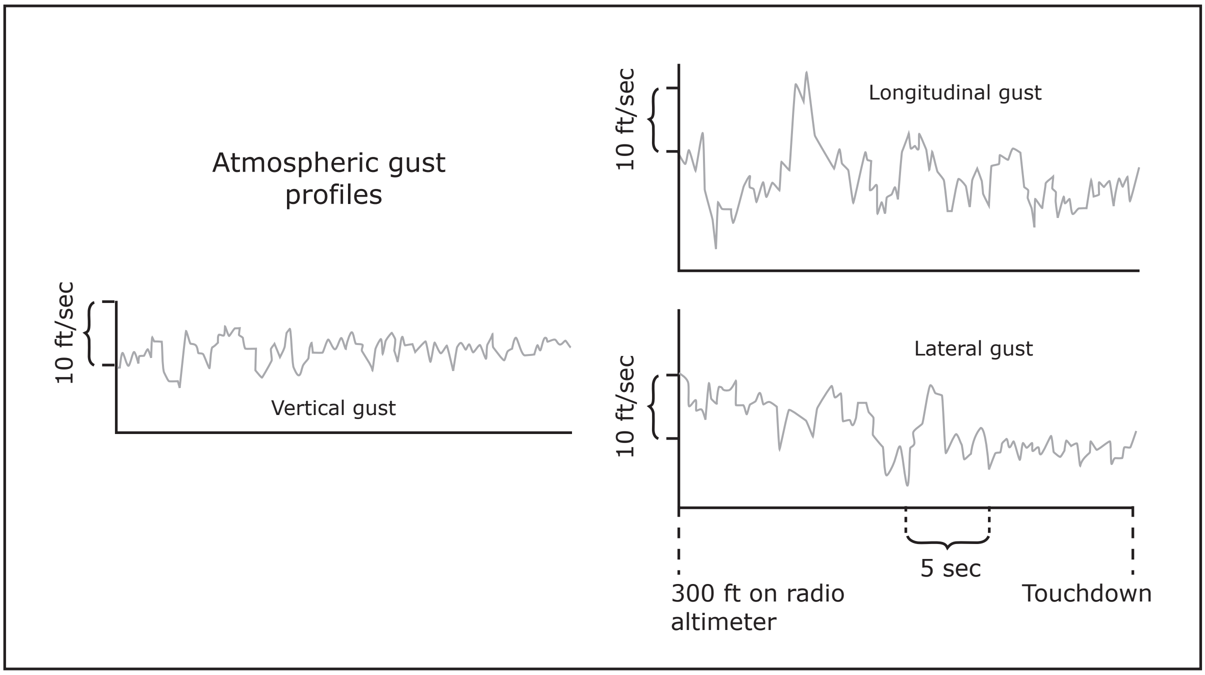

Figure 2: Wind gust examples

Wind Shear

- **Wind shear** is of special interest because it involves significant local changes in the vertical and horizontal velocity (e.g. downdraft)
  – Particularly important near airports during landings.
- Simple analysis – consider the case shown, where the change in (horizontal) wind velocity is represented by

$$u_g = \left(\frac{du}{dh}\right)\Delta h$$

where

  – Δh corresponds to changes in altitude (what we previously just called h)
  – Value du/dh gives magnitude of *wind shear*, which is 0.08–0.15s^{-1} for moderate and 0.15–0.2s^{-1} for strong.

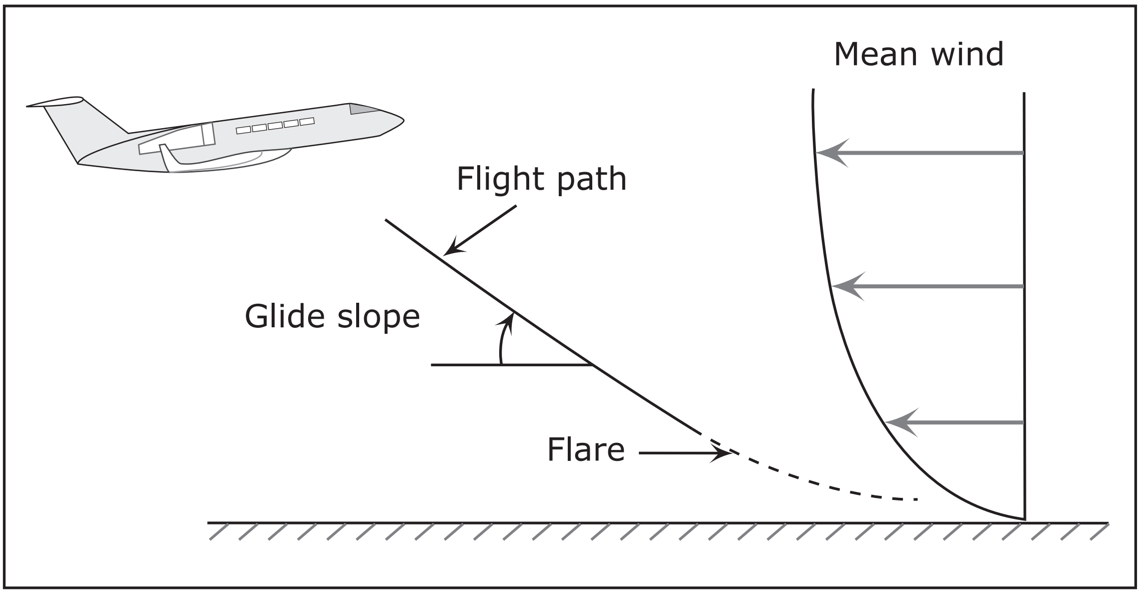

Figure 3: Aircraft descending into a horizontal windshear

- But can write the changes in altitude (use $\dot{h} = \Delta h/\Delta t$) as

$$\dot{h} = U_0(\theta - \alpha)$$

- Now have a coupled situation that is quite interesting
  - Forces on the aircraft change with altitude (h) because height changes the gust velocity
  - Changes in the forces on the aircraft impact the height of the airplane since both θ and α will change.

- Analyze the coupling by looking at the longitudinal equations

$$\begin{aligned} x &= \begin{bmatrix} u & w & q & \theta & h \end{bmatrix}^T \\ h &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} x = C_h x \end{aligned}$$

with controls fixed (zero) and input u_g

$$\begin{aligned} \dot{x} = \tilde{A}x + \tilde{B}_w(:,1)u_g &= \tilde{A}x + \tilde{B}_w(:,1)\frac{du}{dh}h \\ &= \tilde{A}x + \frac{du}{dh}\tilde{B}_w(:,1)C_h x \\ &= \left(\tilde{A} + \frac{du}{dh}\tilde{B}_w(:,1)C_h\right)x \end{aligned}$$

$\Rightarrow$ dynamics have been modified because of the coupling between the change in altitude and the change in forces (with u_g).

- Closed this loop on the B747 dynamics to obtain

$\frac{du}{dh}$	Phugoid Poles
0	-0.0033 $\pm$ 0.0672**i**
0.08	-0.0014 $\pm$ 0.1150**i**
0.15	0.0002 $\pm$ 0.1442**i**
0.2	0.0014 $\pm$ 0.1619**i**

  - Clearly this coupling is not good, and an unstable Phugoid mode is to be avoided during landing operations.

Wind Code

```
% Gust modeling
% 16.333, Fall 2004
% Jonathan P. How

Xu=-1.982e3;Xw=4.025e3;Zu=-2.595e4;Zw=-9.030e4;Zq=-4.524e5;Zwd=1.909e3;
Mu=1.593e4;Mw=-1.563e5;Mq=-1.521e7;Mwd=-1.702e4;
%
g=9.81;theta0=0;S=511;cbar=8.324;
U0=235.9;Iyy=.449e8;m=2.83176e6/g;cbar=8.324;rho=0.3045;
Xdp=.3*m*g;Zdp=0;Mdp=0;
Xde=-3.818e-6*(1/2*rho*U0^2*S);Zde=-0.3648*(1/2*rho*U0^2*S);
Mde=-1.444*(1/2*rho*U0^2*S*cbar);;
%
Ehat=[m 0 0 0;0 m-Zwd 0 0;0 -Mwd Iyy 0;0 0 0 1];
Ahat=[Xu Xw 0 -m*g*cos(theta0);[Zu Zw Zq+m*U0 -m*g*sin(theta0)];
   [Mu Mw Mq 0];[ 0 0 1 0]];
Bhat=[Xde Xdp;Zde Zdp;Mde Mdp;0 0];
%
% form the gust input matrix
Bwhat=[-Xu -Xw 0;-Zu -Zw 0;-Mu -Mw -Mq; 0 0 0];
%
% add height state
Ehat(5,5)=1; % \dot h  state not coupled
Ahat(5,5)=0;Ahat(5,[1:4])=[0 -1 0 U0]; % add height state
Bhat(5,2)=0; % noinput
Bwhat(5,3)=0; % no input

% form the full model
% E \dot x = \hat A x + \hat B u_controls + \hat B_w w
% ==> \dot x = A x + B u_controls + B_w w
%
A=inv(Ehat)*Ahat;
B=inv(Ehat)*Bhat;
Bw=inv(Ehat)*Bwhat;
%
% set u_controls=0
% assume that w_g=q_g=0 and
% u_g = (du/dh) h
%
du_dh=[0 .08 .15 .2];
A1=A+Bw(:,1)*[0 0 0 0 1]*du_dh(1);
A2=A+Bw(:,1)*[0 0 0 0 1]*du_dh(2);
A3=A+Bw(:,1)*[0 0 0 0 1]*du_dh(3);
A4=A+Bw(:,1)*[0 0 0 0 1]*du_dh(4);

ev1=eig(A1);ev2=eig(A2);ev3=eig(A3);ev4=eig(A4);
plot([ev1 ev2 ev3 ev4],'x')
[ev1 ev2 ev3 ev4]
```