Matrix Diagonalization

- Suppose A is diagonizable with independent eigenvectors

$$V = [v_1, \ldots, v_n]$$

  – use similarity transformations to diagonalize dynamics matrix

$$\begin{aligned} \dot{x} &= Ax \Rightarrow \dot{x}_d = A_d x_d \\ V^{-1}AV &= \begin{bmatrix} \lambda_1 & & \\ & \cdot & \\ & & \lambda_n \end{bmatrix} \triangleq \Lambda = A_d \end{aligned}$$

  – Corresponds to change of state from x to $x_d = V^{-1}x$

- System response given by e^{At}, look at power series expansion

$$\begin{aligned} At &= V\Lambda t V^{-1} \\ (At)^2 &= (V\Lambda t V^{-1})V\Lambda t V^{-1} = V\Lambda^2 t^2 V^{-1} \\ \Rightarrow (At)^n &= V\Lambda^n t^n V^{-1} \\ e^{At} &= I + At + \frac{1}{2}(At)^2 + \ldots \\ &= V\left\{I + \Lambda + \frac{1}{2}\Lambda^2 t^2 + \ldots\right\}V^{-1} \\ &= Ve^{\Lambda t}V^{-1} = V\begin{bmatrix} e^{\lambda_1 t} & & \\ & \cdot & \\ & & e^{\lambda_n t} \end{bmatrix}V^{-1} \end{aligned}$$

- Taking Laplace transform,

$$
\begin{aligned}
(sI - A)^{-1} &= V \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \cdot & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} V^{-1} \\
&= \sum_{i=1}^{n} \frac{R_i}{s - \lambda_i}
\end{aligned}
$$

where the residue $R_i = v_i w_i^T$, and we define

$$
V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} \quad , V^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}
$$

- Note that the w_i are the left eigenvectors of A associated with the right eigenvectors v_i

$$
AV = V \begin{bmatrix} \lambda_1 & & \\ & \cdot & \\ & & \lambda_n \end{bmatrix} \Rightarrow V^{-1} A = \begin{bmatrix} \lambda_1 & & \\ & \cdot & \\ & & \lambda_n \end{bmatrix} V^{-1}
$$

$$
\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} A = \begin{bmatrix} \lambda_1 & & \\ & \cdot & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}
$$

where $w_i^T A = \lambda_i w_i^T$

- So, if $\dot{x} = Ax$, the time domain solution is given by

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T x(0) \qquad \text{dyad} \\ x(t) &= \sum_{i=1}^{n} [w_i^T x(0)] e^{\lambda_i t} v_i \end{aligned}$$

- The part of the solution $v_i e^{\lambda_i t}$ is called a **mode** of a system
  - solution is a weighted sum of the system modes
  - weights depend on the components of $x(0)$ along w_i

- Can now give dynamics interpretation of left and right eigenvectors:

$$A v_i = \lambda_i v_i \quad , w_i A = \lambda_i w_i \quad , w_i^T v_j = \delta_{ij}$$

so if $x(0) = v_i$, then

$$\begin{aligned} x(t) &= \sum_{i=1}^{n} (w_i^T x(0)) e^{\lambda_i t} v_i \\ &= e^{\lambda_i t} v_i \end{aligned}$$

$\Rightarrow$ so **right** eigenvectors are initial conditions that result in relatively simple motions $x(t)$.

With no external inputs, if the initial condition only disturbs one mode, then the response consists of only that mode for all time.

- If A has complex conjugate eigenvalues, the process is similar but a little more complicated.
- Consider a 2x2 case with A having eigenvalues $a \pm b\mathbf{i}$ and associated eigenvectors e_1, e_2, with $e_2 = \bar{e}_1$. Then

$$\begin{aligned} A &= \left[\, e_1 \,\middle|\, e_2 \,\right] \begin{bmatrix} a + b\mathbf{i} & 0 \\ 0 & a - b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, e_2 \,\right]^{-1} \\ &= \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right] \begin{bmatrix} a + b\mathbf{i} & 0 \\ 0 & a - b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right]^{-1} \equiv TDT^{-1} \end{aligned}$$

- Now use the transformation matrix

$$M = 0.5 \begin{bmatrix} 1 & -\mathbf{i} \\ 1 & \mathbf{i} \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 1 \\ \mathbf{i} & -\mathbf{i} \end{bmatrix}$$

- Then it follows that

$$\begin{aligned} A &= TDT^{-1} = (TM)(M^{-1}DM)(M^{-1}T^{-1}) \\ &= (TM)(M^{-1}DM)(TM)^{-1} \end{aligned}$$

which has the nice structure:

$$A = \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1}$$

where all the matrices are real.

- With complex roots, the diagonalization is to a block diagonal form.

- For this case we have that

$$e^{At} = \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1}$$

- Note that $\left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1}$ is the matrix that inverts $\left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]$

$$\left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1} \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- So for an initial condition to excite just this mode, can pick $x(0) = [\texttt{Re}(e_1)]$, or $x(0) = [\texttt{Im}(e_1)]$ or a linear combination.

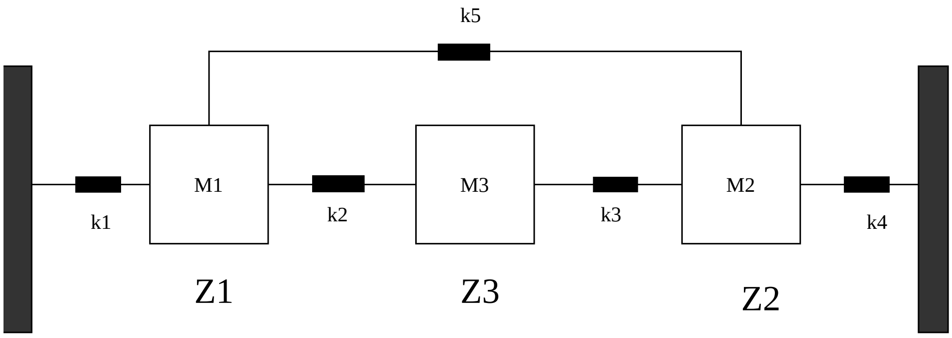
- Example $x(0) = [\texttt{Re}(e_1)]$

$$\begin{aligned} x(t) &= e^{At}x(0) = \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \cdot \\ &\qquad \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right]^{-1} [\texttt{Re}(e_1)] \\ &= \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= e^{at} \left[\, \texttt{Re}(e_1) \,\middle|\, \texttt{Im}(e_1) \,\right] \begin{bmatrix} \cos(bt) \\ -\sin(bt) \end{bmatrix} \\ &= e^{at} \left(\texttt{Re}(e_1)\cos(bt) - \texttt{Im}(e_1)\sin(bt)\right) \end{aligned}$$

which would ensure that only this mode is excited in the response

Example: Spring Mass System

- Classic example: spring mass system consider simple case first: $m_i = 1$, and $k_i = 1$



$$
\begin{aligned}
x &= \begin{bmatrix} z_1 & z_2 & z_3 & \dot{z}_1 & \dot{z}_2 & \dot{z}_3 \end{bmatrix} \\
A &= \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \quad M = \mathbf{diag}\,(m_i) \\
K &= \begin{bmatrix} k_1 + k_2 + k_5 & -k_5 & -k_2 \\ -k_5 & k_3 + k_4 + k_5 & -k_3 \\ -k_2 & -k_3 & k_2 + k_3 \end{bmatrix}
\end{aligned}
$$

- Eigenvalues and eigenvectors of the undamped system

$$\lambda_1 = \pm 0.77\mathbf{i} \quad \lambda_2 = \pm 1.85\mathbf{i} \quad \lambda_3 = \pm 2.00\mathbf{i}$$

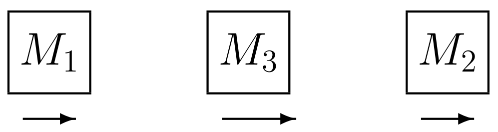
v_1	v_2	v_3
1.00	1.00	1.00
1.00	1.00	-1.00
1.41	-1.41	0.00
$\pm 0.77\mathbf{i}$	$\pm 1.85\mathbf{i}$	$\pm 2.00\mathbf{i}$
$\pm 0.77\mathbf{i}$	$\pm 1.85\mathbf{i}$	$\mp 2.00\mathbf{i}$
$\pm 1.08\mathbf{i}$	$\mp 2.61\mathbf{i}$	0.00

- Initial conditions to excite just the three modes:

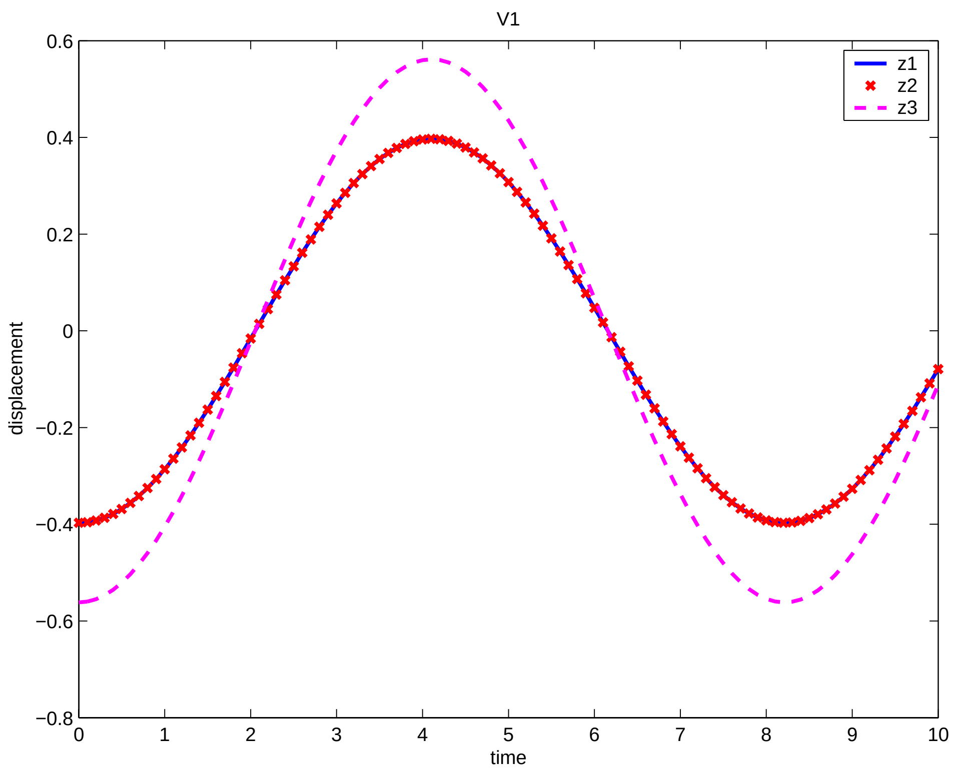
$$x_i(0) = \alpha_1 \mathrm{Re}(v_i) + \alpha_2 \mathrm{Im}(v_1) \quad \forall \alpha_j \in \mathbf{R}$$

  – Simulation using $\alpha_1 = 1$, $\alpha_2 = 0$

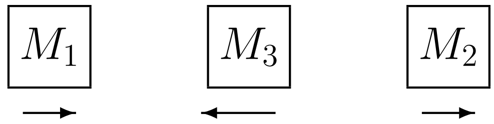
- Visualization important for correct physical interpretation
- Mode 1 $\lambda_1 = \pm 0.77\mathbf{i}$



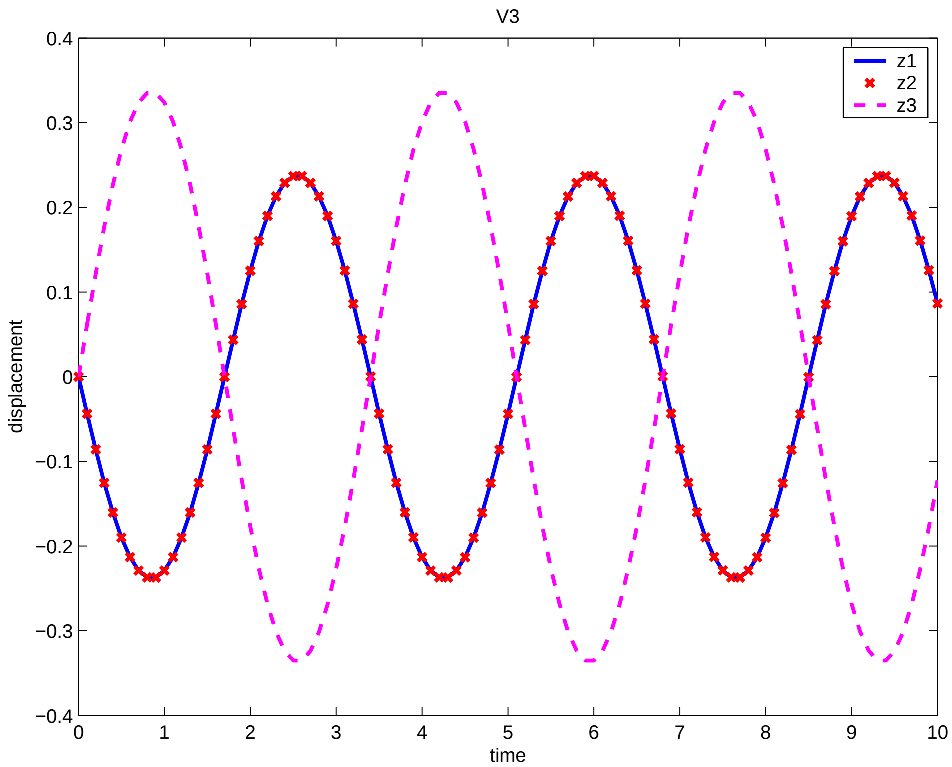
  – Lowest frequency mode, all masses move in same direction
  – Middle mass has higher amplitude motions z_3, motions all in phase



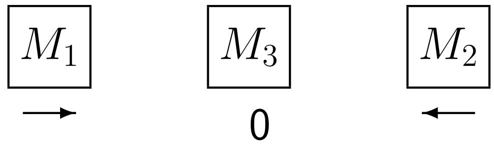
- Mode 2 $\lambda_2 = \pm 1.85\mathbf{i}$



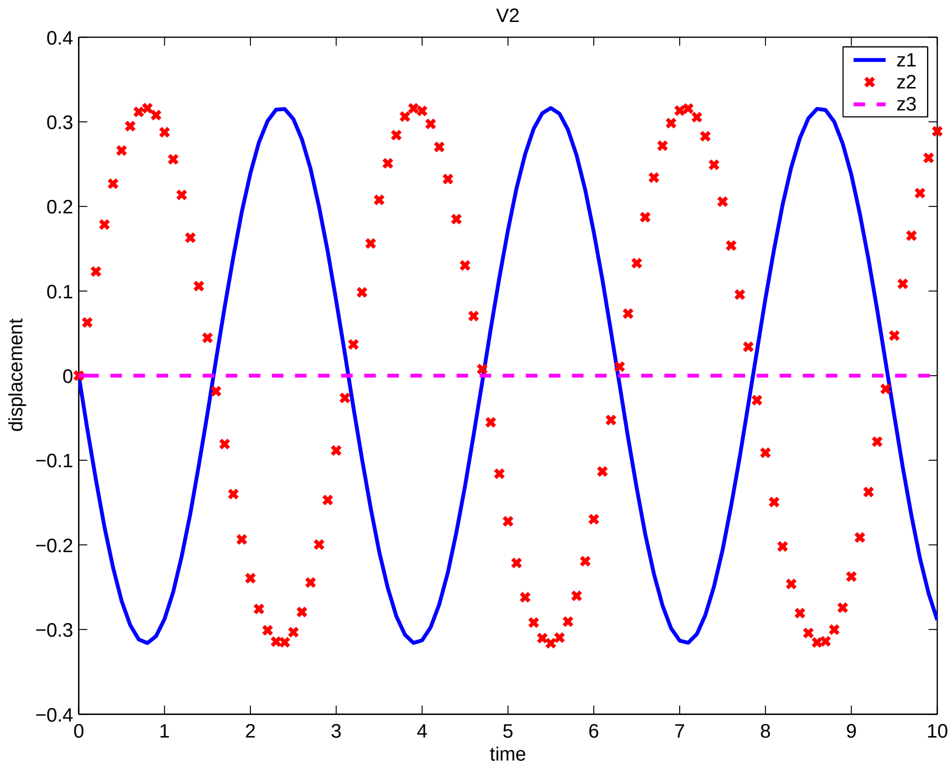
  – Middle frequency mode has middle mass moving in opposition to two end masses
  – Again middle mass has higher amplitude motions z_3



- Mode 3 $\lambda_3 = \pm 2.00\mathbf{i}$



  - Highest frequency mode, has middle mass stationary, and other two masses in opposition



- Eigenvectors with that correspond with more constrained motion of the system are associated with higher frequency eigenvalues