16.323 Principles of Optimal Control
Spring 2008

16.323 Lecture 8

Properties of Optimal Control Solution

Bryson and Ho – Section 3.5 and Kirk – Section 4.4

Properties of Optimal Control

- If $\mathbf{g} = \mathbf{g}(\mathbf{x}, \mathbf{u})$ and $\mathbf{a} = \mathbf{a}(\mathbf{x}, \mathbf{u})$ do not explicitly depend on time t, then the Hamiltonian H is at least piecewise constant.

$$H = g(\mathbf{x}, \mathbf{u}) + \mathbf{p}^T \mathbf{a}(\mathbf{x}, \mathbf{u}) \tag{8.1}$$

then

$$\frac{dH}{dt} = \cancelto{0}{\frac{\partial H}{\partial t}} + \left(\frac{\partial H}{\partial \mathbf{x}}\right)\frac{d\mathbf{x}}{dt} + \left(\frac{\partial H}{\partial \mathbf{u}}\right)\frac{d\mathbf{u}}{dt} + \left(\frac{\partial H}{\partial \mathbf{p}}\right)\frac{d\mathbf{p}}{dt} \tag{8.2}$$

$$= H_{\mathbf{x}}\mathbf{a} + H_{\mathbf{u}}\dot{\mathbf{u}} + H_{\mathbf{p}}\dot{\mathbf{p}} \tag{8.3}$$

Now use the necessary conditions:

$$\dot{\mathbf{x}} = \mathbf{a} = H_{\mathbf{p}}^T \tag{8.4}$$

$$\dot{\mathbf{p}} = -H_{\mathbf{x}}^T \tag{8.5}$$

to get that

$$\frac{dH}{dt} = -\dot{\mathbf{p}}^T\mathbf{a} + \mathbf{a}^T\dot{\mathbf{p}} + H_{\mathbf{u}}\dot{\mathbf{u}} = H_{\mathbf{u}}\dot{\mathbf{u}}$$

- Third necessary condition requires $H_{\mathbf{u}} = 0$, so clearly $\frac{dH}{dt} = 0$, which suggests H is a constant,
  - Note that it might be possible for the value of this constant to change at a discontinuity of $\mathbf{u}$, since then $\dot{\mathbf{u}}$ would be infinite, and $0 \cdot \infty$ is not defined.
  - Thus H is at least piecewise constant

- For free final time problems, transversality condition gives,

$$h_t + H(t_f) = 0.$$

  - If h is not a function of time, then $h_t = 0$ so $H(t_f) = 0$
  - With no jumps in $\mathbf{u}$, H is constant $\Rightarrow$ $H = 0$ for all time.

- If solution has a corner that is not induced by an intermediate state variable constraint, then H, $\mathbf{p}$, and $H_{\mathbf{u}}$ are all continuous across the corner.

- To see, this, write augmented cost functional on 6–1 in the form

$$J = \text{terminal terms} + \int_{t_0}^{t_f} \left(\mathbf{g} + \mathbf{p}^T(\mathbf{a} - \dot{\mathbf{x}})\right) dt$$

and recall definition of Hamiltonian $H = \mathbf{g} + \mathbf{p}^T\mathbf{a}$, so that

$$J = \text{terminal terms} + \int_{t_0}^{t_f} \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right) dt$$

- Looks similar to the classical form analyzed on 5–16

$$\tilde{J} = \int_{t_0}^{t_f} g\left(x, \dot{x}, t\right) dt$$

which led to two Weierstrass-Erdmann corner conditions

$$g_{\dot{\mathbf{x}}}(t_1^-) = g_{\dot{\mathbf{x}}}(t_1^+) \tag{8.6}$$

$$g(t_1^-) - g_{\dot{\mathbf{x}}}(t_1^-)\dot{\mathbf{x}}(t_1^-) = g(t_1^+) - g_{\dot{\mathbf{x}}}(t_1^+)\dot{\mathbf{x}}(t_1^+) \tag{8.7}$$

- With $g\left(x, \dot{x}, t\right) \Rightarrow H - \mathbf{p}^T\dot{\mathbf{x}}$, equivalent continuity conditions are:

$$\frac{\partial(H - \mathbf{p}^T\dot{\mathbf{x}})}{\partial\dot{\mathbf{x}}} = -\mathbf{p}^T \qquad \text{must be cts at corner}$$

and

$$\begin{aligned}(H - \mathbf{p}^T\dot{\mathbf{x}}) - \frac{\partial(H - \mathbf{p}^T\dot{\mathbf{x}})}{\partial\dot{\mathbf{x}}}\dot{\mathbf{x}} &\\ = (H - \mathbf{p}^T\dot{\mathbf{x}}) + \mathbf{p}^T\dot{\mathbf{x}} &\\ = H \qquad \text{must be cts at corner} &\end{aligned} \tag{8.8}$$

- So both $\mathbf{p}(t)$ and H must be continuous across a corner that is not induced by a state variable equality/inequality constraint.

Point State Constraint

- Consider what happens with an interior point state constraint (Bryson, section 3.5) of the form that

$$\mathbf{N}(\mathbf{x}(t_1), t_1) = 0$$

where $t_0 < t_1 < t_f$ and $\mathbf{N}$ is a vector of $q < n$ constraints.
– Assume that $\mathbf{x}(t_0)$, $\mathbf{x}(t_f)$, t_0, and t_f all specified.

- Augment constraint to cost (6–1) using multiplier $\boldsymbol{\pi}$

$$J_a = h(\mathbf{x}(t_f), t_f) + \boldsymbol{\pi}^T\mathbf{N} + \int_{t_0}^{t_f} \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right) dt$$

- Proceed as before with the corner conditions (5–15), and split cost integral into 2 parts

$$\int_{t_0}^{t_f} \Rrightarrow \int_{t_0}^{t_1} + \int_{t_1}^{t_f}$$

and form the variation (drop terms associated with t_0 and t_f):

$$\begin{aligned} \delta J_a = &\mathbf{N}^T(t_1)\delta\boldsymbol{\pi} + \boldsymbol{\pi}^T(\mathbf{N}_\mathbf{x}(t_1)\delta\mathbf{x}_1 + \mathbf{N}_t(t_1)\delta t_1) \\ &+ \int_{t_0}^{t_1} \left(H_\mathbf{x}\delta\mathbf{x} + H_\mathbf{u}\delta\mathbf{u} + (H_\mathbf{p} - \dot{\mathbf{x}}^T)\delta\mathbf{p} - \mathbf{p}^T\delta\dot{\mathbf{x}}\right) dt \\ &+ \int_{t_1}^{t_f} \left(H_\mathbf{x}\delta\mathbf{x} + H_\mathbf{u}\delta\mathbf{u} + (H_\mathbf{p} - \dot{\mathbf{x}}^T)\delta\mathbf{p} - \mathbf{p}^T\delta\dot{\mathbf{x}}\right) dt \\ &+ \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right)\Big|^{t_1^-} \delta t_1 + \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right)\Big|_{t_1^+} \delta t_1 \end{aligned} \tag{8.9}$$

Collect:

$$\begin{aligned} = &\mathbf{N}^T(t_1)\delta\boldsymbol{\pi} + \boldsymbol{\pi}^T(\mathbf{N}_\mathbf{x}(t_1)\delta\mathbf{x}_1 + \mathbf{N}_t(t_1)\delta t_1) \\ &+ \int_{t_0}^{t_1} \left(H_\mathbf{x}\delta\mathbf{x} + H_\mathbf{u}\delta\mathbf{u} + (H_\mathbf{p} - \dot{\mathbf{x}}^T)\delta\mathbf{p} - \mathbf{p}^T\delta\dot{\mathbf{x}}\right) dt \\ &+ \int_{t_1}^{t_f} \left(H_\mathbf{x}\delta\mathbf{x} + H_\mathbf{u}\delta\mathbf{u} + (H_\mathbf{p} - \dot{\mathbf{x}}^T)\delta\mathbf{p} - \mathbf{p}^T\delta\dot{\mathbf{x}}\right) dt \\ &+ \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right)(t_1^-)\delta t_1 - \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right)(t_1^+)\delta t_1 \end{aligned} \tag{8.10}$$

- On 6–2 showed that the IBP will give:

$$
\begin{aligned}
-\int_{t_0}^{t_1} \mathbf{p}^T \delta\dot{\mathbf{x}} dt &= -\mathbf{p}^T(t_1^-)\left(\delta\mathbf{x}_1 - \dot{\mathbf{x}}(t_1^-)\delta t_1\right) + \int_{t_0}^{t_1} \dot{\mathbf{p}}^T \delta\mathbf{x} dt \\
-\int_{t_1}^{t_f} \mathbf{p}^T \delta\dot{\mathbf{x}} dt &= \mathbf{p}^T(t_1^+)\left(\delta\mathbf{x}_1 - \dot{\mathbf{x}}(t_1^+)\delta t_1\right) + \int_{t_1}^{t_f} \dot{\mathbf{p}}^T \delta\mathbf{x} dt
\end{aligned}
$$

- Substitute into (13) to get

$$
\begin{aligned}
\delta J_a = \mathbf{N}^T(t_1)\delta\boldsymbol{\pi} &+ \boldsymbol{\pi}^T(\mathbf{N}_{\mathbf{x}}(t_1)\delta\mathbf{x}_1 + \mathbf{N}_t(t_1)\delta t_1) \qquad (8.11) \\
&+ \int_{t_0}^{t_f} \left((H_{\mathbf{x}} + \dot{\mathbf{p}}^T)\delta\mathbf{x} + H_{\mathbf{u}}\delta\mathbf{u} + (H_{\mathbf{p}} - \dot{\mathbf{x}}^T)\delta\mathbf{p}\right) dt \\
&+ \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right)(t_1^-)\delta t_1 - \left(H - \mathbf{p}^T\dot{\mathbf{x}}\right)(t_1^+)\delta t_1 \\
&- \mathbf{p}^T(t_1^-)\left(\delta\mathbf{x}_1 - \dot{\mathbf{x}}(t_1^-)\delta t_1\right) + \mathbf{p}^T(t_1^+)\left(\delta\mathbf{x}_1 - \dot{\mathbf{x}}(t_1^+)\delta t_1\right)
\end{aligned}
$$

- Rearrange and cancel terms

$$
\begin{aligned}
\delta J_a = \mathbf{N}^T(t_1)\delta\boldsymbol{\pi} &+ \int_{t_0}^{t_f} \left((H_{\mathbf{x}} + \dot{\mathbf{p}}^T)\delta\mathbf{x} + H_{\mathbf{u}}\delta\mathbf{u} + (H_{\mathbf{p}} - \dot{\mathbf{x}}^T)\delta\mathbf{p}\right) dt \\
&+ \left[\mathbf{p}^T(t_1^+) - \mathbf{p}^T(t_1^-) + \boldsymbol{\pi}^T\mathbf{N}_{\mathbf{x}}(t_1)\right]\delta\mathbf{x}_1 \qquad (8.12) \\
&+ \left[H(t_1^-) - H(t_1^+) + \boldsymbol{\pi}^T\mathbf{N}_t(t_1)\right]\delta t_1
\end{aligned}
$$

- So choose $H(t_1^-)$ & $H(t_1^+)$ and $\mathbf{p}^T(t_1^-)$ & $\mathbf{p}^T(t_1^+)$ to ensure that the coefficients of $\delta\mathbf{x}_1$ and t_1 vanish in (15), giving:

$$
\begin{aligned}
\mathbf{p}^T(t_1^-) &= \mathbf{p}^T(t_1^+) + \boldsymbol{\pi}^T\mathbf{N}_{\mathbf{x}}(t_1) \\
H(t_1^-) &= H(t_1^+) - \boldsymbol{\pi}^T\mathbf{N}_t(t_1)
\end{aligned}
$$

- These explicitly show that $\mathbf{p}(t_1)$ and $H(t_1)$ are discontinuous across the state constraint induced corner, but $H_{\mathbf{u}}$ will be continuous.