16.323 Principles of Optimal Control
Spring 2008

# 16.323 Lecture 15

Signals and System Norms

$\mathcal{H}_\infty$ Synthesis

Different type of optimal controller


**SP** Skogestad and Postlethwaite(1996) Multivariable Feedback Control Wiley.

**JB** Burl (2000). Linear Optimal Control Addison-Wesley.

**ZDG** Zhou, Doyle, and Glover (1996). Robust and Optimal Control Prentice Hall.

**MAC** Maciejowski (1989) Multivariable Feedback Design Addison Wesley.

# Mathematical Background

- **Signal norms** we use norms to measure the size of a signal.
  - Three key properties of a norm:
    1. $\|u\| \geq 0$, and $\|u\| = 0$ iff $u = 0$
    2. $\|\alpha u\| = |\alpha|\|u\|$ $\forall$ scalars $\alpha$
    3. $\|u + v\| \leq \|u\| + \|v\|$

- Key signal norms
  - 2-norm of $u(t)$ – *Energy of the signal*

$$\|u(t)\|_2 \equiv \left[\int_{-\infty}^{\infty} u^2(t) dt\right]^{1/2}$$

  - $\infty$-norm of $u(t)$ – *maximum value over time*

$$\|u(t)\|_\infty = \max_t |u(t)|$$

  - Other useful measures include the *Average power*

$$\text{pow}(u(t)) = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u^2(t) dt\right)^{1/2}$$

    $u(t)$ is called a *power signal* if $\text{pow}(u(t)) < \infty$

- **System norms** Consider the system with dynamics $y = G(s)u$
  – Assume $G(s)$ stable, LTI transfer function matrix
  – $g(t)$ is the associated impulse response matrix (causal).

- $\mathcal{H}_2$ norm for the system: (LQG problem)

$$\begin{aligned}\|G\|_2 &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \texttt{trace}[G^H(j\omega)G(j\omega)]d\omega\right)^{1/2} \\ &= \left(\int_0^{\infty} \texttt{trace}[g^T(\tau)g(\tau)]d\tau\right)^{1/2}\end{aligned}$$

  Two interpretations:
  – For SISO: energy in the output $y(t)$ for a unit impulse input $u(t)$.
  – For MIMO [27]: apply an impulsive input separately to each actuator and measure the response $z_i$, then

$$\|G\|_2^2 = \sum_i \|z_i\|_2^2$$

  – Can also interpret as the expected RMS value of the output in response to unit-intensity white noise input excitation.

- **Key point:** Can show that

$$\|G\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_i \sigma_i^2[G(j\omega)]d\omega\right)^{1/2}$$

  – Where $\sigma_i[G(j\omega)]$ is the $i$th singular value[28] [29] of the system $G(s)$ evaluated at $s = \mathbf{j}\omega$
  – $\mathcal{H}_2$ norm concerned with **overall performance** ($\sum_i \sigma_i^2$) **over all** frequencies

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[27] ZDG114

[28] http://mathworld.wolfram.com/SingularValueDecomposition.html

[29] http://en.wikipedia.org/wiki/Singular_value_decomposition

- $\mathcal{H}_\infty$ norm for the system:

$$\|G(s)\|_\infty = \sup_\omega \overline{\sigma}[G(\mathrm{j}\omega)]$$

Interpretation:

– $\|G(s)\|_\infty$ is the "energy gain" from the input $u$ to output $y$

$$\|G(s)\|_\infty = \max_{u(t)\neq 0} \frac{\int_0^\infty y^T(t)y(t)dt}{\int_0^\infty u^T(t)u(t)dt}$$

– Achieve this maximum gain using a **worst case** input signal that is essentially a sinusoid at frequency $\omega^\star$ with input direction that yields $\overline{\sigma}[G(\mathrm{j}\omega^\star)]$ as the amplification.

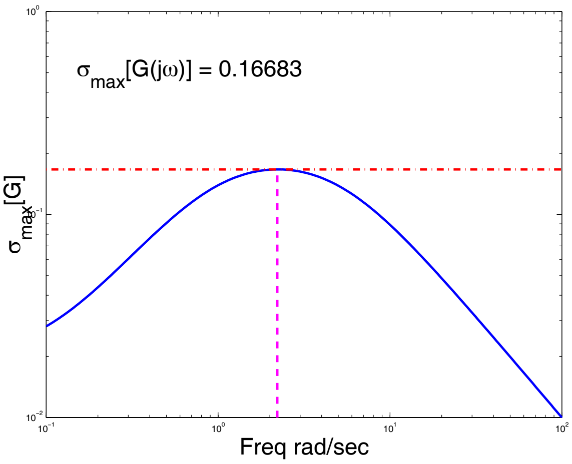



Figure 15.1: Graphical test for the $\|G\|_\infty$.

- Note that we now have
  1. Signal norm $\|u(t)\|_\infty = \max_t |u(t)|$
  2. Vector norm $\|x\|_\infty = \max_i |x_i|$
  3. System norm $\|G(s)\|_\infty = \max_\omega \overline{\sigma}[G(\mathrm{j}\omega)]$

  We use the same symbol $\|\cdot\|_\infty$ for all three, but there is typically no confusion, as the norm to be used is always clear by the context.

# Key Points / Summary

- So $\mathcal{H}_\infty$ is concerned primarily with the **peaks** in the frequency response, and the $\mathcal{H}_2$ norm is concerned with the **overall** response.

- The $\mathcal{H}_\infty$ norm satisfies the **submultiplicative property**

$$\|GH\|_\infty \leq \|G\|_\infty \cdot \|H\|_\infty$$

  – Will see that this is an essential property for the robustness tests

  – **Does not hold** in general for $\|GH\|_2$

- Reference to $\mathcal{H}_\infty$ **control** is that we would like to design a stabilizing controller that ensures that the peaks in the transfer function matrix of interest are *knocked down*.

$$\text{e.g. want} \quad \max_\omega \overline{\sigma}[T(\mathrm{j}\omega)] \equiv \|T(s)\|_\infty < 0.75$$

- Reference to $\mathcal{H}_2$ **control** is that we would like to design a stabilizing controller that reduces the $\|T(s)\|_2$ as much as possible.

  – Note that $\mathcal{H}_2$ control and LQG are the same thing.

# Computation

- Assume that $G(s) = C(sI - A)^{-1}B + D$ with $\mathcal{R}\lambda(A) < 0$, i.e. $G(s)$ stable.

- $\mathcal{H}_2$ norm: requires a strictly proper system $D = 0$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

  – Define:
    **Observability Gramian** $P_o$

$$A^T P_o + P_o A + C^T C = 0 \Leftrightarrow P_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

    **Controllability Gramian** $P_c$

$$A P_c + P_c A^T + B B^T = 0 \Leftrightarrow P_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

  then

$$\|G\|_2^2 = \texttt{trace}\left(B^T P_o B\right) = \texttt{trace}\left(C P_c C^T\right)$$

  **Proof:** use the impulse response of the system $G(s)$ and evaluate the time-domain version of the norm.

- $\mathcal{H}_\infty$ norm: Define the **Hamiltonian matrix**

$$H = \left[\begin{array}{c|c} A + B(\gamma^2 I - D^T D)^{-1} D^T C & B(\gamma^2 I - D^T D)^{-1} B^T \\ \hline -C^T(I + D(\gamma^2 I - D^T D)^{-1} D^T) C & -(A + B(\gamma^2 I - D^T D)^{-1} D^T C)^T \end{array}\right]$$

  – Then $\|G(s)\|_\infty < \gamma$ iff $\overline{\sigma}(D) < \gamma$ and $H$ has no eigenvalues on the **j**$\omega$-axis.
  – Graphical test $\max_\omega \overline{\sigma}[G(\mathrm{j}\omega)] < \gamma$ replaced with eigenvalue test.

# Issues

- Note that it is **not easy** to find $\|G\|_\infty$ directly using the state space techniques
  - It is easy to check if $\|G\|_\infty < \gamma$
  - So we just keep changing $\gamma$ to find the smallest value for which we can show that $\|G\|_\infty < \gamma$ (called $\gamma_{\min}$)

  $\Rightarrow$ Bisection search algorithm.

- **Bisection search algorithm**
  1. Select $\gamma_u$, $\gamma_l$ so that $\gamma_l \leq \|G\|_\infty \leq \gamma_u$
  2. Test $(\gamma_u - \gamma_l)/\gamma_l < \text{TOL}$.
     **Yes** $\Rightarrow$ Stop ($\|G\|_\infty \approx \frac{1}{2}(\gamma_u + \gamma_l)$)
     **No** $\Rightarrow$ go to step 3.
  3. With $\gamma = \frac{1}{2}(\gamma_l + \gamma_u)$, test if $\|G\|_\infty < \gamma$ using $\lambda_i(H)$
  4. If $\lambda_i(H) \in \mathbf{j}\mathcal{R}$, then set $\gamma_l = \gamma$ (test value too low), otherwise set $\gamma_u = \gamma$ and go to step 2.

# Application

- Note that we can use the state space tests to analyze the weighted tests that we developed for robust stability
  - For example, we have seen the value in ensuring that the sensitivity remains smaller than a particular value

$$\overline{\sigma}[W_i S(j\omega)] < 1 \ \ \forall \ \ \omega$$

- We can test this by determining if $\|W_i(s)S(s)\|_\infty < 1$
  - Use state space models of $G_c(s)$ and $G(s)$ to develop a state space model of

$$S(s) := \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & 0 \end{array}\right]$$

  - Augment these dynamics with the (stable, min phase) $W_i(s)$ to get a model of $W_i(s)S(s)$

$$W_i(s) \ := \ \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & 0 \end{array}\right]$$

$$W_i(s)S(s) \ := \ \left[\begin{array}{cc|c} A_s & 0 & B_s \\ B_w C_s & A_w & 0 \\ \hline 0 & C_w & 0 \end{array}\right]$$

  - Now compute the $\mathcal{H}_\infty$ norm of the combined system $W_i(s)S(s)$.

- Note that, with $D = 0$, the $\mathcal{H}_\infty$ **Hamiltonian matrix** becomes

$$H = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

  - Know that $\|G\|_\infty < \gamma$ iff $H$ has no eigenvalues on the **j**$\omega$-axis.
  - Equivalent test is if there exists a $X \geq 0$ such that

$$A^TX + XA + C^TC + \frac{1}{\gamma^2}XBB^TX = 0$$

    and $A + \frac{1}{\gamma^2}BB^TX$ is stable.
  - So there is a direction relationship between the Hamiltonian matrix $H$ and the **algebraic Riccati Equation** (ARE)

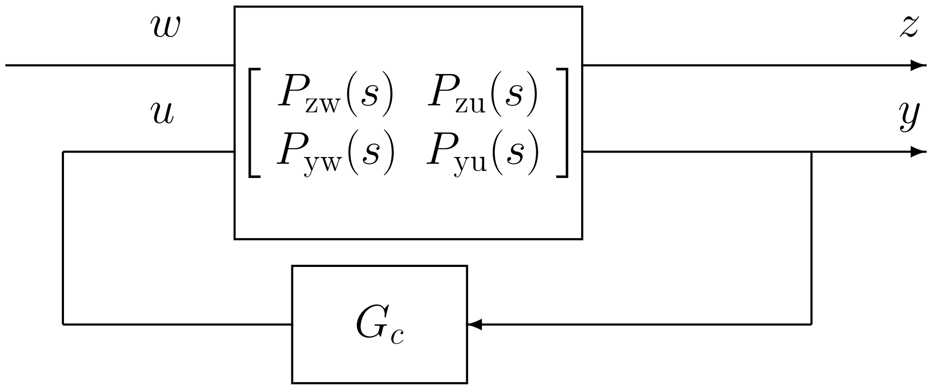
- **Aside:** Compare this ARE with the one that we would get if we used this system in an LQR problem:

$$A^TP + PA + C^TC - \frac{1}{\rho}PBB^TP = 0$$

  - If $(A, B, C)$ stabilizable/detectable, then will always get a solution for the LQR ARE.
  - Sign difference in quadratic term of the $\mathcal{H}_\infty$ ARE makes this equation harder to satisfy. Consistent with the fact that we could have $\|G\|_\infty > \gamma \Rightarrow$ no solution to the $\mathcal{H}_\infty$ ARE.
  - The two Riccati equations look similar, but with the sign change, the solutions can behave **very** differently.

# Synthesis

- For the synthesis problem, we typically define a generalized version of the system dynamics




**Signals:**

- $z$ Performance output
- $w$ Disturbance/ref inputs
- $y$ Sensor outputs
- $u$ Actuator inputs

**Generalized plant:**

$$P(s) = \begin{bmatrix} P_{\rm zw}(s) & P_{\rm zu}(s) \\ P_{\rm yw}(s) & P_{\rm yu}(s) \end{bmatrix}$$

contains the plant $G(s)$ and all performance and uncertainty weights

- With the loop closed ($u = G_c y$), can show that

$$\begin{aligned} \left(\frac{z}{w}\right)_{CL} &= P_{\rm zw} + P_{\rm zu} G_c (I - P_{\rm yu} G_c)^{-1} P_{\rm yw} \\ &\equiv F_l(P, G_c) \end{aligned}$$

called a (lower) **Linear Fractional Transformation** (LFT).

- **Design Objective:** Find $G_c(s)$ to stabilize the closed-loop system and minimize $\|F_l(P, G_c)\|_\infty$.

- Hard problem to solve, so we typically consider a suboptimal problem:
  - Find $G_c(s)$ to satisfy $\|F_l(P, G_c)\|_\infty < \gamma$
  - Then use bisection (called a $\gamma$ **iteration**) to find the smallest value ($\gamma_{opt}$) for which $\|F_l(P, G_c)\|_\infty < \gamma_{opt}$

    $\Rightarrow$ hopefully get that $G_c$ approaches $G_c^{\text{opt}}$

- Consider the suboptimal $\mathcal{H}_\infty$ synthesis problem: [30]

$$\text{Find } G_c(s) \text{ to satisfy } \|F_l(P, G_c)\|_\infty < \gamma$$

$$P(s) = \begin{bmatrix} P_{\text{zw}}(s) & P_{\text{zu}}(s) \\ P_{\text{yw}}(s) & P_{\text{yu}}(s) \end{bmatrix} := \left[\begin{array}{c|cc} A & B_{\text{w}} & B_{\text{u}} \\ \hline C_{\text{z}} & 0 & D_{\text{zu}} \\ C_{\text{y}} & D_{\text{yw}} & 0 \end{array}\right]$$

where we assume that:

1. $(A, B_{\text{u}}, C_{\text{y}})$ is stabilizable/detectable (essential)
2. $(A, B_{\text{w}}, C_{\text{z}})$ is stabilizable/detectable (essential)
3. $D_{\text{zu}}^T [\, C_{\text{z}} \;\; D_{\text{zu}} \,] = [\, 0 \;\; I \,]$ (simplify/essential)
4. $\begin{bmatrix} B_{\text{w}} \\ D_{\text{yw}} \end{bmatrix} D_{\text{yw}}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (simplify/essential)

- Note that we will not cover all the details of the solution to this problem – it is well covered in the texts.

[30] SP367

- There exists a stabilizing $G_c(s)$ such that $\|F_l(P, G_c)\|_\infty < \gamma$ iff

  $$(1) \;\; \exists X \geq 0 \;\; \text{that solves the ARE}$$

  $$A^T X + XA + C_{\rm z}^T C_{\rm z} + X(\gamma^{-2} B_{\rm w} B_{\rm w}^T - B_{\rm u} B_{\rm u}^T) X = 0$$

  and $\mathcal{R}\lambda_i \left[A + (\gamma^{-2} B_{\rm w} B_{\rm w}^T - B_{\rm u} B_{\rm u}^T) X\right] < 0 \;\; \forall\, i$

  $$(2) \;\; \exists Y \geq 0 \;\; \text{that solves the ARE}$$

  $$AY + YA^T + B_{\rm w}^T B_{\rm w} + Y(\gamma^{-2} C_{\rm z}^T C_{\rm z} - C_{\rm y}^T C_{\rm y}) Y = 0$$

  and $\mathcal{R}\lambda_i \left[A + Y(\gamma^{-2} C_{\rm z}^T C_{\rm z} - C_{\rm y}^T C_{\rm y})\right] < 0 \;\; \forall\, i$

  $$(3) \;\; \rho(XY) < \gamma^2$$

  $\rho$ is the **spectral radius** ($\rho(A) = \max_i |\lambda_i(A)|$).

- Given these solutions, the **central $\mathcal{H}_\infty$ controller** is given by

  $$G_c(s) := \left[\begin{array}{c|c} A + (\gamma^{-2} B_{\rm w} B_{\rm w}^T - B_{\rm u} B_{\rm u}^T) X - ZY C_{\rm y}^T C_{\rm y} & ZY C_{\rm y}^T \\ \hline -B_{\rm u}^T X & 0 \end{array}\right]$$

  where $Z = (I - \gamma^{-2} YX)^{-1}$

  – Central controller has as many states as the generalized plant.

- Note that this design does not decouple as well as the regulator/estimator for LQG

# Observations

- Basic assumptions:

(A1) $(A, B_{\rm u}, C_{\rm y})$ is stabilizable/detectable
(A2) $(A, B_{\rm w}, C_{\rm z})$ is stabilizable/detectable
(A3) $D_{\rm zu}^T[\ C_{\rm z}\ \ D_{\rm zu}\ ] = [\ 0\ \ I\ ]$ (scaling and no cross-coupling)
(A4) $\begin{bmatrix} B_{\rm w} \\ D_{\rm yw} \end{bmatrix} D_{\rm yw}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (scaling and no cross-coupling)

- The restrictions that $D_{\rm zw} = 0$ and $D_{\rm yu} = 0$ are weak, and can easily be removed (the codes handle the more general $D$ case).

- (A1) is required to ensure that it is even possible to get a stabilizing controller.

- Need $D_{\rm zu}$ and $D_{\rm yw}$ to have full rank to ensure that we penalize control effort (A3) and include sensor noise (A4)
  ⇒ Avoids singular case with infinite bandwidth controllers.
  ⇒ Often where you will have the most difficulties initially.

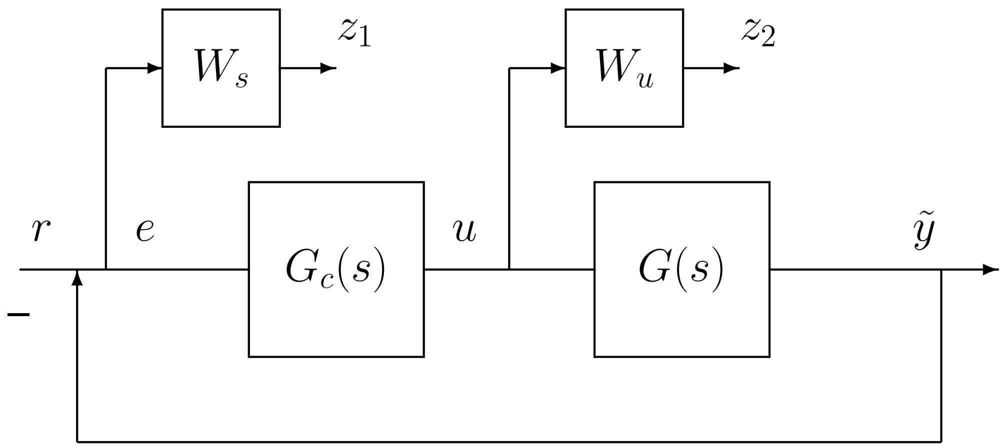
- Typically will see two of the assumptions written as:

(Ai) $\begin{bmatrix} A - \mathbf{j}\omega I & B_{\rm u} \\ C_{\rm z} & D_{\rm zu} \end{bmatrix}$ has full column rank $\forall\ \ \omega$

(Aii) $\begin{bmatrix} A - \mathbf{j}\omega I & B_{\rm w} \\ C_{\rm y} & D_{\rm yw} \end{bmatrix}$ has full row rank $\forall\ \ \omega$

  – These ensure that there are **no** **j**$\omega$-axis zeros in the $P_{\rm zu}$ or $P_{\rm yw}$ TF's – cannot have the controller canceling these, because that design would not internally stabilize the closed-loop system.
  – But with assumptions (A3) and (A4) given above, can show that A(i) and A(ii) are equivalent to our assumption (A2).

# Simple Design Example




where

$$G = \frac{200}{(0.05s + 1)^2(10s + 1)}$$

- Note that we have 1 input ($r$) and two performance outputs - one that penalizes the sensitivity $S(s)$ of the system, and the other that penalizes the control effort used.

- Easy to show (see next page) that the closed-loop is:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_s S \\ W_u G_c S \end{bmatrix} r$$

  where, in this case, the input $r$ acts as the "disturbance input" $w$ to the generalized system.

- To achieve good low frequency tracking and a crossover frequency of about 10 rad/sec, pick

$$W_s = \frac{s/1.5 + 10}{s + (10) \cdot (0.0001)} \qquad W_u = 1$$

- Generalized system in this case:

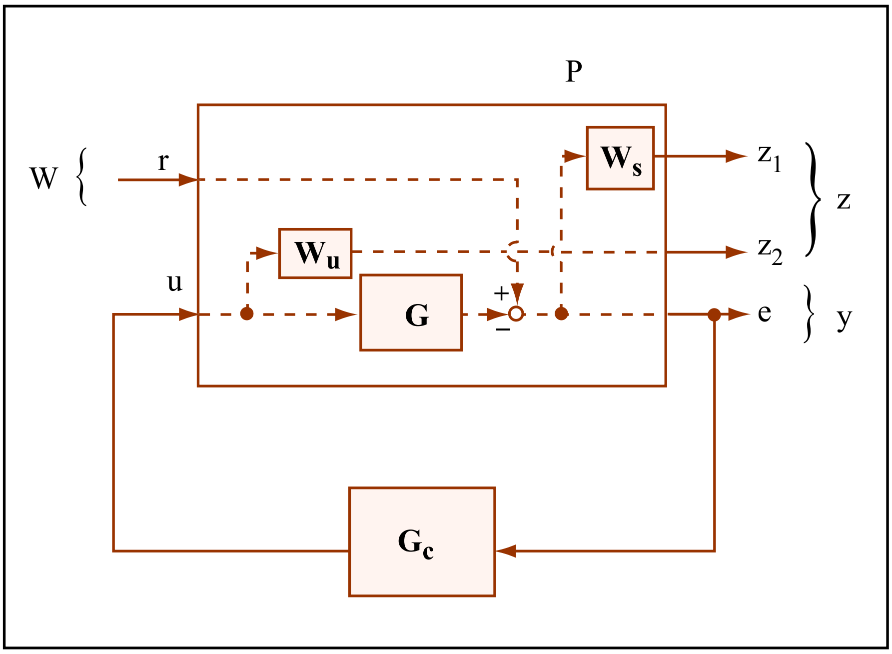



Figure by MIT OpenCourseWare.

Figure 15.2: Rearrangement of original picture in the generalized plant format.

- Derive $P(s)$ as

$$\begin{aligned} z_1 &= W_s(s)(r - Gu) \\ z_2 &= W_u u \\ e &= r - Gu \\ u &= G_c e \end{aligned}$$

$$\begin{aligned} P(s) &= \left[\begin{array}{c|c} W_s(s) & -W_s(s)G(s) \\ 0 & W_u(s) \\ \hline 1 & -G(s) \end{array}\right] \\ &= \begin{bmatrix} P_{\text{zw}}(s) & P_{\text{zu}}(s) \\ P_{\text{yw}}(s) & P_{\text{yu}}(s) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} P_{CL} &= F_l(P, G_c) \\ &= \begin{bmatrix} W_s \\ 0 \end{bmatrix} + \begin{bmatrix} -W_s G \\ W_u \end{bmatrix} G_c(I + GG_c)^{-1} 1 \\ &= \begin{bmatrix} W_s - W_s G G_c S \\ W_u G_c S \end{bmatrix} = \begin{bmatrix} W_s S \\ W_u G_c S \end{bmatrix} \end{aligned}$$

- In state space form, let

$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad W_s(s) := \left[\begin{array}{c|c} A_w & B_{\rm w} \\ \hline C_w & D_w \end{array}\right] \quad W_u = 1$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{x}_w &= A_w x_w + B_{\rm w} e = A_w x_w + B_{\rm w} r - B_{\rm w} C x \\ z_1 &= C_w x_w + D_w e = C_w x_w + D_w r - D_w C x \\ z_2 &= W_u u \\ e &= r - Cx \end{aligned}$$

$$P(s) := \left[\begin{array}{cc|c|c} A & 0 & 0 & B \\ -B_{\rm w} C & A_w & B_{\rm w} & 0 \\ \hline -D_w C & C_w & D_w & 0 \\ 0 & 0 & 0 & W_u \\ \hline -C & 0 & 1 & 0 \end{array}\right]$$

- Now use the mu-tools code to solve for the controller. (Could also have used the robust control toolbox code).

```
A=[Ag zeros(n1,n2);-Bsw*Cg Asw];
Bw=[zeros(n1,1);Bsw];
Bu=[Bg;zeros(n2,1)];
Cz=[-Dsw*Cg Csw;zeros(1,n1+n2)];
Cy=[-Cg zeros(1,n2)];
Dzw=[Dsw;0];
Dzu=[0;1];
Dyw=[1];
Dyu=0;
P=pck(A,[Bw Bu],[Cz;Cy],[Dzw Dzu;Dyw Dyu]);
% call hinf to find Gc (mu toolbox)
[Gc,G,gamma]=hinfsyn(P,1,1,0.1,20,.001);
```

- Results from the $\gamma$-iteration showing whether we pass or fail the various $X$, $Y$, $\rho(XY)$ tests as we keep searching over $\gamma$, starting at the initial bound of 20.

```
Resetting value of Gamma min based on D_11, D_12, D_21 terms

Test bounds:      0.6667 <  gamma  <=     20.0000

    gamma    hamx_eig  xinf_eig  hamy_eig   yinf_eig   nrho_xy   p/f
     20.000  9.6e+000  6.2e-008  1.0e-003   0.0e+000    0.0000    p
     10.333  9.6e+000  6.3e-008  1.0e-003   0.0e+000    0.0000    p
      5.500  9.5e+000  6.3e-008  1.0e-003   0.0e+000    0.0000    p
      3.083  9.5e+000  6.5e-008  1.0e-003   0.0e+000    0.0000    p
      1.875  9.4e+000  6.9e-008  1.0e-003   0.0e+000    0.0000    p
>>    1.271  9.1e+000  -1.2e+004# 1.0e-003   -4.5e-010    0.0000    f
      1.573  9.3e+000  7.3e-008  1.0e-003   0.0e+000    0.0000    p
      1.422  9.2e+000  7.6e-008  1.0e-003   0.0e+000    0.0000    p
>>    1.346  9.2e+000  -6.4e+004# 1.0e-003   0.0e+000    0.0000    f
      1.384  9.2e+000  7.7e-008  1.0e-003   0.0e+000    0.0000    p
>>    1.365  9.2e+000  -1.9e+006# 1.0e-003   0.0e+000    0.0000    f
      1.375  9.2e+000  7.7e-008  1.0e-003   -4.5e-010    0.0000    p
      1.370  9.2e+000  7.7e-008  1.0e-003   0.0e+000    0.0000    p
      1.368  9.2e+000  7.7e-008  1.0e-003   0.0e+000    0.0000    p
      1.366  9.2e+000  7.7e-008  1.0e-003   0.0e+000    0.0000    p
>>    1.366  9.2e+000  -1.3e+007# 1.0e-003   0.0e+000    0.0000    f

 Gamma value achieved:     1.3664
```

- Since $\gamma_{\min} = 1.3664$, this indicates that we **did not** meet the desired goal of $|S| < 1/|W_s|$ (can only say that $|S| < 1.3664/|W_s|$).
  - Confirmed by the plot, which shows that we just fail the test (blue line passes above magenta)

- But note that, even though this design fails the sensitivity weight - we still get pretty good performance
  - For performance problems, can think of the objective of getting $\gamma_{\min} < 1$ as a "design goal" $\rightsquigarrow$ it is "not crucial"
  - Use $W_u$ to tune the control design

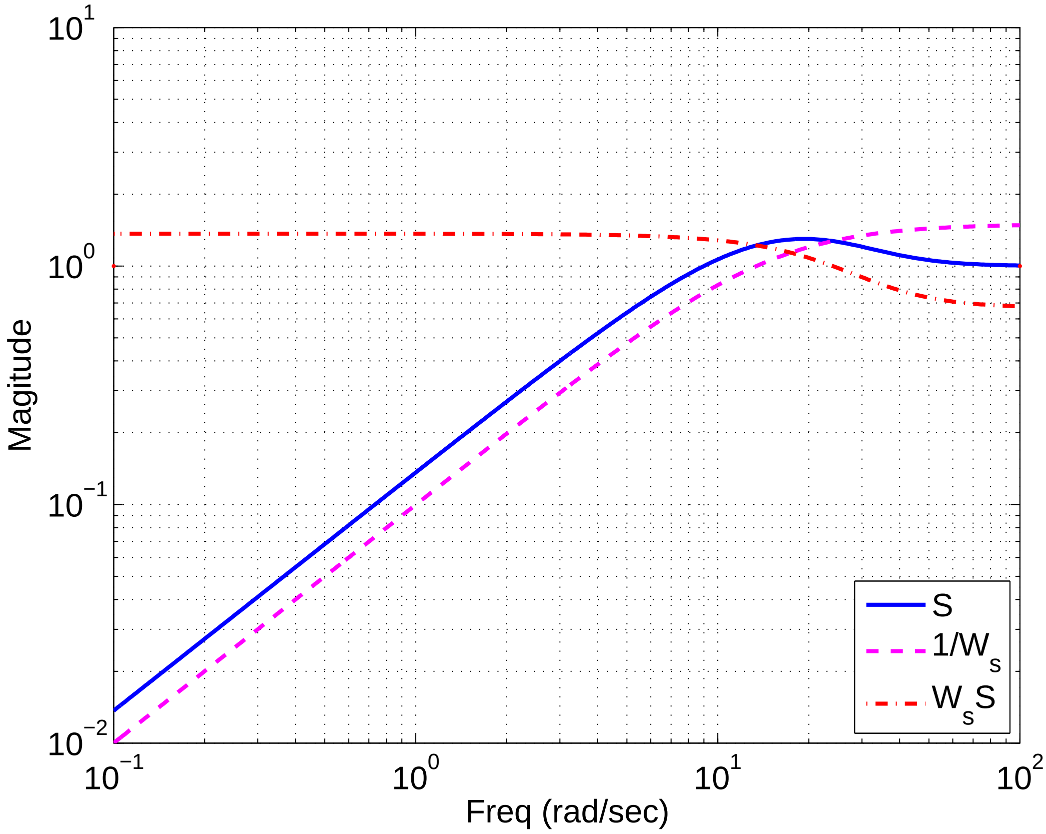



Figure 15.3: Visualization of the weighted sensitivity tests.

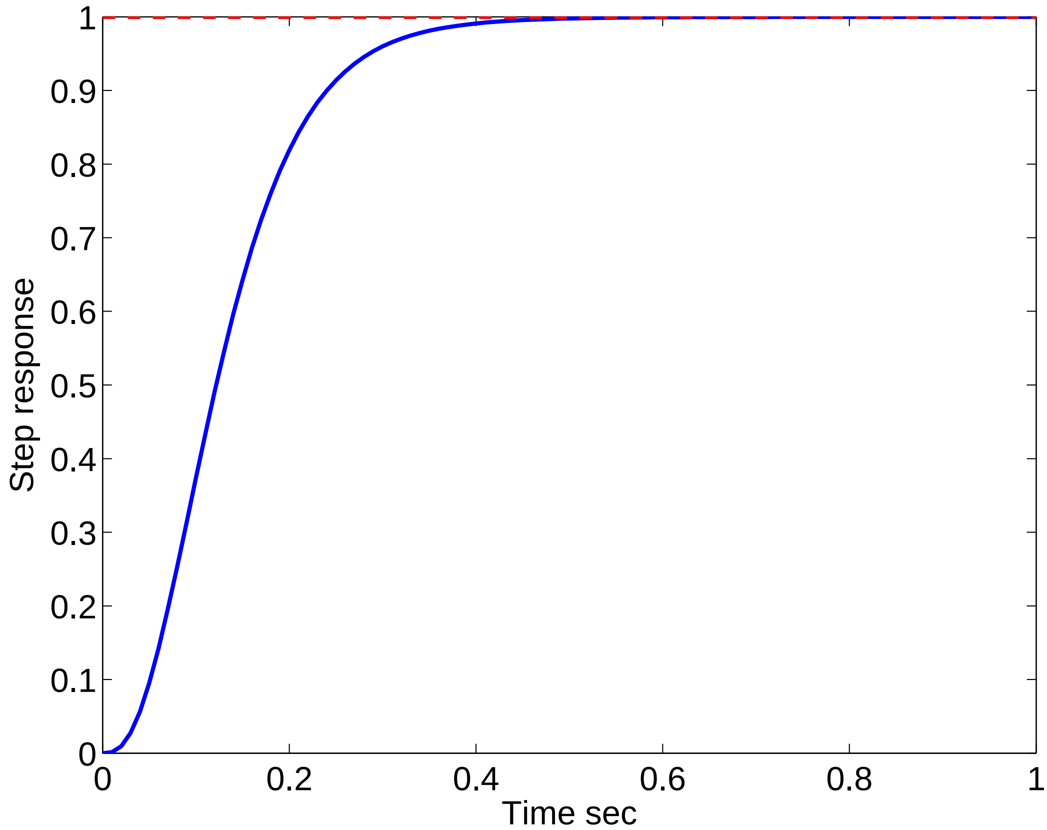



Figure 15.4: Time response of controller that yields $\gamma_{\min} = 1.3664$.

# General LQG Problem

- Can also put LQG ($\mathcal{H}_2$) design into this generalized framework [31].
- Define the dynamics

$$\begin{aligned} \dot{x} &= Ax + Bu + w_d \\ y &= Cx + w_n \end{aligned}$$

where

$$E\left\{\begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix}\begin{bmatrix} w_d^T(\tau) & w_n^T(\tau) \end{bmatrix}\right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}\delta(t-\tau)$$

- LQG problem is to find controller $u = G_c(s)y$ that minimizes

$$J = E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T (x^T R_{xx} x + u^T R_{uu} u)dt\right\}$$

- To put this problem in the general framework, define

$$z = \begin{bmatrix} R_{xx}^{1/2} & 0 \\ 0 & R_{uu}^{1/2} \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} w_d \\ w_n \end{bmatrix} = \begin{bmatrix} W^{1/2} & 0 \\ 0 & V^{1/2} \end{bmatrix} w$$

where $w$ is a unit intensity white noise process.

- With $z = F_l(P, G_c)w$, the LQG cost function can be rewritten as

$$J = E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T z^T(t)z(t)dt\right\} = \|F_l(P,G_c)\|_2^2$$

- In this case the generalized plant matrix is

$$P(s) := \left[\begin{array}{c|cc|c} A & W^{1/2} & 0 & B \\ \hline R_{xx}^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{uu}^{1/2} \\ \hline C & 0 & V^{1/2} & 0 \end{array}\right]$$

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# Controller Interpretations

- Given these solutions, the **central $\mathcal{H}_\infty$ controller** is given by

$$G_c(s) := \left[\begin{array}{c|c} A + (\gamma^{-2}B_{\rm w}B_{\rm w}^T - B_{\rm u}B_{\rm u}^T)X - ZYC_{\rm y}^TC_{\rm y} & ZYC_{\rm y}^T \\ \hline -B_{\rm u}^TX & 0 \end{array}\right]$$

where $Z = (I - \gamma^{-2}YX)^{-1}$

- Can develop a further interpretation of this controller if we rewrite the dynamics as:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + \gamma^{-2}B_{\rm w}B_{\rm w}^TX\hat{x} - B_{\rm u}B_{\rm u}^TX\hat{x} - ZYC_{\rm y}^TC_{\rm y}\hat{x} + ZYC_{\rm y}^Ty \\ u &= -B_{\rm u}^TX\hat{x} \end{aligned}$$

$$\Rightarrow \dot{\hat{x}} = A\hat{x} + B_{\rm w}\left[\gamma^{-2}B_{\rm w}^TX\hat{x}\right] + B_{\rm u}\left[-B_{\rm u}^TX\hat{x}\right] + ZYC_{\rm y}^T\left[y - C_{\rm y}\hat{x}\right]$$

$$\Rightarrow \dot{\hat{x}} = A\hat{x} + B_{\rm w}\left[\gamma^{-2}B_{\rm w}^TX\hat{x}\right] + B_{\rm u}u + L\left[y - C_{\rm y}\hat{x}\right]$$

looks very similar to Kalman Filter developed for LQG controller.

- The difference is that we have an additional input $\hat{w}_{\rm worst} = \gamma^{-2}B_{\rm w}^TX\hat{x}$ that enters through $B_{\rm w}$.
  – $w_{\rm worst}$ is an estimate of **worst-case** disturbance to the system.

- Finally, note that a separation rule does exist for the $\mathcal{H}_\infty$ controller. But we will not discuss it.

# Code: $\mathcal{H}_\infty$ Synthesis

```
% Hinf example
% 16.323 MIT Spring 2007
% Jon How
%
set(0,'DefaultAxesFontName','arial')
set(0,'DefaultAxesFontSize',16)
set(0,'DefaultTextFontName','arial')
set(0,'DefaultTextFontSize',20)

clear all
if ~exist('yprev')
   yprev=[1 1]';
   tprev=[0 1]';
   Sensprev=[1 1];
   fprev=[.1 100];
end

%Wu=1/1e9;
Wu=1;
% define plant
[Ag,Bg,Cg,Dg]=tf2ss(200,conv(conv([0.05 1],[0.05 1]),[10 1]));
Gol=ss(Ag,Bg,Cg,Dg);
% define sensitivity weight
M=1.5;wB=10;A=1e-4;
[Asw,Bsw,Csw,Dsw]=tf2ss([1/M wB],[1 wB*A]);
Ws=ss(Asw,Bsw,Csw,Dsw);
% form augmented P dynamics
n1=size(Ag,1);
n2=size(Asw,1);
A=[Ag zeros(n1,n2);-Bsw*Cg Asw];
Bw=[zeros(n1,1);Bsw];
Bu=[Bg;zeros(n2,1)];
Cz=[-Dsw*Cg Csw;zeros(1,n1+n2)];
Cy=[-Cg zeros(1,n2)];
Dzw=[Dsw;0];
Dzu=[0;Wu];
Dyw=[1];
Dyu=0;
P=pck(A,[Bw Bu],[Cz;Cy],[Dzw Dzu;Dyw Dyu]);

% call hinf to find Gc (mu toolbox)
diary hinf1_diary
[Gc,G,gamma]=hinfsyn(P,1,1,0.1,20,.001);
diary off

[ac,bc,cc,dc]=unpck(Gc);
ev=max(real(eig(ac)/2/pi))

PP=ss(A,[Bw Bu],[Cz;Cy],[Dzw Dzu;Dyw Dyu]);
GGc=ss(ac,bc,cc,dc);
CLsys = feedback(PP,GGc,[2],[3],1);
[acl,bcl,ccl,dcl]=ssdata(CLsys);
% reduce closed-loop system so that it only has
% 1 input and 2 outputs
bcl=bcl(:,1);ccl=ccl([1 2],:);dcl=dcl([1 2],1);
CLsys=ss(acl,bcl,ccl,dcl);

f=logspace(-1,2,400);
Pcl=freqresp(CLsys,f);
CLWS=squeeze(Pcl(1,1,:)); % closed loop weighted sens
WS=freqresp(Ws,f); % sens weight
SensW=squeeze(WS(1,1,:));
Sens=CLWS./SensW; % divide out weight to get closed-loop sens
figure(1);clf
loglog(f,abs(Sens),'b-','LineWidth',2)
hold on
loglog(f,abs(1./SensW),'m--','LineWidth',2)
```

```
loglog(f,abs(CLWS),'r-.','LineWidth',2)
loglog(fprev,abs(Sensprev),'r.')
legend('S','1/W_s','W_sS','Location','SouthEast')
hold off
xlabel('Freq (rad/sec)')
ylabel('Magitude')
grid

print -depsc hinf1.eps;jpdf('hinf1')

na=size(Ag,1);
nac=size(ac,1);
Acl=[Ag Bg*cc;-bc*Cg ac];Bcl=[zeros(na,1);bc];Ccl=[Cg zeros(1,nac)];Dcl=0;
Gcl=ss(Acl,Bcl,Ccl,Dcl);
[y,t]=step(Gcl,1);

figure(2);clf
plot(t,y,'LineWidth',2)
hold on;plot(tprev,yprev,'r--','LineWidth',2);hold off
xlabel('Time sec')
ylabel('Step response')

print -depsc hinf12.eps;jpdf('hinf12')

yprev=y;
tprev=t;
Sensprev=Sens;
fprev=f;
```