16.323 Principles of Optimal Control
Spring 2008

Topic #14

16.31 Feedback Control Systems

MIMO Systems

- Singular Value Decomposition

Multivariable Frequency Response

- In the MIMO case, the system $G(s)$ is described by a $p \times m$ **transfer function matrix** (TFM)
  - Still have that $G(s) = C(sI - A)^{-1}B + D$
  - But $G(s) \to A, B, C, D$ MUCH less obvious than in SISO case.
  - Also seen that the discussion of poles and zeros of MIMO systems is much more complicated.

- In SISO case we use the Bode plot to develop a measure of the system "size".
  - Given $z = Gw$, where $G(\mathrm{j}\omega) = |G(\mathrm{j}\omega)|e^{j\phi(w)}$
  - Then $w = |w|e^{j(\omega_1 t + \psi)}$ applied to $|G(\mathrm{j}\omega)|e^{j\phi(w)}$ yields

$$|w||G(\mathrm{j}\omega_1)|e^{j(\omega_1 t + \psi + \phi(\omega_1))} = |z|e^{j(\omega_1 t + \psi_o)} \equiv z$$

  - Amplification and phase shift of the input signal obvious in the SISO case.

- MIMO extension?
  - Is the response of the system large or small?

$$G(s) = \begin{bmatrix} 10^3/s & 0 \\ 0 & 10^{-3}/s \end{bmatrix}$$

- For MIMO systems, cannot just plot all of the g_{ij} elements of G
  - Ignores the coupling that might exist between them.
  - So not enlightening.

- **Basic MIMO frequency response:**
  - Restrict all inputs to be at the same frequency
  - Determine how the system responds at that frequency
  - See how this response changes with frequency

- So inputs are $\mathbf{w} = \mathbf{w}_c e^{j\omega t}$, where $\mathbf{w}_c \in \mathbb{C}^m$
  - Then we get $\mathbf{z} = G(s)|_{s=\mathrm{j}\omega}\,\mathbf{w}$, $\quad \Rightarrow \mathbf{z} = \mathbf{z}_c e^{j\omega t}$ and $\mathbf{z}_c \in \mathbb{C}^p$
  - We need only analyze $\mathbf{z}_c = G(\mathrm{j}\omega)\mathbf{w}_c$

- As in the SISO case, we need a way to establish if the system response is **large** or **small**.
  - How much amplification we can get with a bounded input.

- Consider $\mathbf{z}_c = G(\mathrm{j}\omega)\mathbf{w}_c$ and set $\|\mathbf{w}_c\|_2 = \sqrt{\mathbf{w}_c^H \mathbf{w}_c} \leq 1$. What can we say about the $\|\mathbf{z}_c\|_2$?
  - Answer depends on ω and on the **direction** of the input $\mathbf{w}_c$
  - Best found using **singular values.**

Singular Value Decomposition

- Must perform **SVD** of the matrix $G(s)$ at each frequency $s = \mathbf{j}\omega$

$$G(\mathrm{j}\omega) \in \mathbb{C}^{p\times m} \quad U \in \mathbb{C}^{p\times p} \quad \Sigma \in \mathbb{R}^{p\times m} \quad V \in \mathbb{C}^{m\times m}$$

$$G = U\Sigma V^H$$

  – $U^H U = I$, $UU^H = I$, $V^H V = I$, $VV^H = I$, and Σ is diagonal.

  – Diagonal elements $\sigma_k \geq 0$ of Σ are the **singular values** of G.

$$\sigma_i = \sqrt{\lambda_i(G^H G)} \quad \text{or} \quad \sigma_i = \sqrt{\lambda_i(GG^H)}$$

  the positive ones are the same from both formulas.

  – Columns of matrices U and V (u_i and v_j) are the associated eigenvectors

$$\begin{aligned} G^H G v_j &= \sigma_j^2 v_j \\ GG^H u_i &= \sigma_i^2 u_i \\ G v_i &= \sigma_i u_i \end{aligned}$$

- If the rank$(G) = r \leq \min(p, m)$, then
  – $\sigma_k > 0,\ k = 1, \ldots, r$
  – $\sigma_k = 0,\ k = r+1, \ldots, \min(p, m)$
  – Singular values are sorted so that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$

- An SVD gives a **very detailed description** of how a matrix (the system G) acts on a vector (the input $\mathbf{w}$) at a particular frequency.

- So how can we use this result?
  - Fix the size $\|\mathbf{w}_c\|_2 = 1$ of the input, and see how large we can make the output.
  - Since we are working at a single frequency, we just analyze the relation

$$\mathbf{z}_c = G_w \mathbf{w}_c, \qquad G_w \equiv G(s = \mathbf{j}\omega)$$

- Define the maximum and minimum amplifications as:

$$\begin{aligned} \overline{\sigma} &\equiv \max_{\|\mathbf{w}_c\|_2 = 1} \|\mathbf{z}_c\|_2 \\ \underline{\sigma} &\equiv \min_{\|\mathbf{w}_c\|_2 = 1} \|\mathbf{z}_c\|_2 \end{aligned}$$

- Then we have that (let $q = \min(p, m)$)

$$\begin{aligned} \overline{\sigma} &= \sigma_1 \\ \underline{\sigma} &= \begin{cases} \sigma_q & p \geq m \quad \text{“tall”} \\ 0 & p < m \quad \text{“wide”} \end{cases} \end{aligned}$$

- Can use $\overline{\sigma}$ and $\underline{\sigma}$ to determine the possible amplification and attenuation of the input signals.

- Since $G(s)$ changes with frequency, so will $\overline{\sigma}$ and $\underline{\sigma}$

SVD Example

- Consider (wide case)

$$G_w = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= U\Sigma V^H$$

so that $\sigma_1 = 5$ and $\sigma_2 = 0.5$

$$\overline{\sigma} \equiv \max_{\|\mathbf{w}_c\|_2=1} \|G_w \mathbf{w}_c\|_2 = 5 = \sigma_1$$

$$\underline{\sigma} \equiv \min_{\|\mathbf{w}_c\|_2=1} \|G_w \mathbf{w}_c\|_2 = 0 \neq \sigma_2$$

- But now consider (tall case)

$$\tilde{G}_w = \begin{bmatrix} 5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= U\Sigma V^H$$

so that $\sigma_1 = 5$ and $\sigma_2 = 0.5$ still.

$$\overline{\sigma} \equiv \max_{\|\mathbf{w}_c\|_2=1} \|G_w \mathbf{w}_c\|_2 = 5 = \sigma_1$$

$$\underline{\sigma} \equiv \min_{\|\mathbf{w}_c\|_2=1} \|G_w \mathbf{w}_c\|_2 = 0.5 = \sigma_2$$

- For MIMO systems, the gains (or σ's) are only part of the story, as we must also consider the **input direction**.

- To analyze this point further, note that we can rewrite

$$G_w = U\Sigma V^H = \begin{bmatrix} u_1 & \dots & u_p \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_m \\ & 0 & \end{bmatrix} \begin{bmatrix} v_1^H \\ \vdots \\ v_m^H \end{bmatrix}$$

$$= \sum_{i=1}^{m} \sigma_i u_i v_i^H$$

  – Assumed tall case for simplicity, so $p > m$ and $q = m$

- Can now analyze impact of **various alternatives for the input**
  – Only looking at one frequency, so the basic signal is harmonic.
  – But, we are free to pick the relative **sizes** and **phases** of each of the components of the input vector $\mathbf{w}_c$.
    ◇ These define the **input direction**

- For example, we could pick $\mathbf{w}_c = v_1$, then

$$\mathbf{z}_c = G_w \mathbf{w}_c = \left(\sum_{i=1}^{m} \sigma_i u_i v_i^H \right) v_1 = \sigma_1 u_1$$

  since $v_i^H v_j = \delta_{ij}$.

  – Output amplified by σ_1. The relative **sizes** and **phases** of each of the components of the output are given by the vector $\mathbf{z}_c$.

- By selecting other input directions (at the same frequency), we can get quite different amplifications of the input signal

$$\underline{\sigma} \leq \frac{\|G_w \mathbf{w}_c\|_2}{\|\mathbf{w}_c\|_2} \leq \overline{\sigma}$$

- Thus we say that
  – G_w is large if $\underline{\sigma}(G_w) \gg 1$

  – G_w is small if $\overline{\sigma}(G_w) \ll 1$

- **MIMO frequency response** are plots of $\overline{\sigma}(j\omega)$ and $\underline{\sigma}(j\omega)$.
  – Then use the singular value vectors to analyze the response at a particular frequency.