16.323 Principles of Optimal Control
Spring 2008

16.323 Lecture 13

LQG Robustness

- Stengel Chapter 6
- Question: how well do the large gain and phase margins discussed for LQR (6–29) map over to LQG?

LQG

- When we use the combination of an optimal estimator and an optimal regulator to design the controller, the compensator is called

  Linear Quadratic Gaussian (LQG)

  – Special case of the controllers that can be designed using the separation principle.

- The great news about an LQG design is that stability of the closed-loop system is **guaranteed**.
  – The designer is freed from having to perform any detailed mechanics - the entire process is fast and can be automated.

- So the designer can focus on the "performance" related issues, being confident that the LQG design will produce a controller that stabilizes the system.
  – How to specify the state cost function (i.e. selecting $\mathbf{z} = C_z \mathbf{x}$) and what values of R_{zz}, R_{uu} to use.
  – Determine how the process and sensor noise enter into the system and what their relative sizes are (i.e. select R_{ww} & R_{vv})

- This sounds great – so what is the catch??

- The remaining issue is that sometimes the controllers designed using these state-space tools are very sensitive to errors in the knowledge of the model.
  – *i.e.,* the compensator might work **very well** if the plant gain $\alpha = 1$, but be unstable if it is $\alpha = 0.9$ or $\alpha = 1.1$.
  – LQG is also prone to plant–pole/compensator–zero cancelation, which tends to be sensitive to modeling errors.
  – J. Doyle, "Guaranteed Margins for LQG Regulators", IEEE *Transactions on Automatic Control*, Vol. 23, No. 4, pp. 756-757, 1978.

- The good news is that the state-space techniques will give you a controller very easily.
  - **You should use the time saved to verify that the one you designed is a "good" controller.**

- There are, of course, different definitions of what makes a controller **good**, but one important criterion is whether **there is a reasonable chance that it would work on the real system as well as it does in Matlab.** $\Rightarrow$ **Robustness**.
  - The controller must be able to tolerate some modeling error, because our models in Matlab are typically inaccurate.
    - ◇ Linearized model
    - ◇ Some parameters poorly known
    - ◇ Ignores some higher frequency dynamics

- Need to develop tools that will give us some insight on how well a controller can tolerate modeling errors.

- Consider the "cart on a stick" system, with the dynamics as given in the following pages. Define

$$q = \begin{bmatrix} \theta \\ x \end{bmatrix} \ , \quad \mathbf{x} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

Then with $y = x$

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B_u u \\ y &= C_y \mathbf{x} \end{aligned}$$

- For the parameters given in the notes, the system has an unstable pole at $+5.6$ and one at $s = 0$. There are plant zeros at ± 5.

- Very simple LQG design - main result is fairly independent of the choice of the weighting matrices.

- The resulting compensator is unstable (+23!!)
  – This is somewhat expected. (why?)

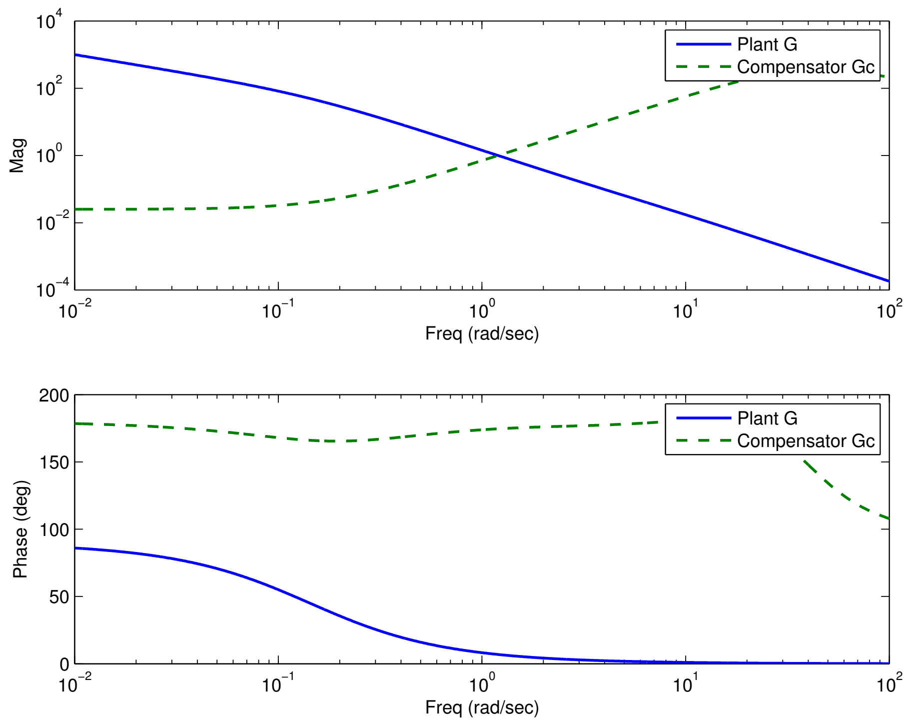


Figure 13.1: Plant and Controller

Example: cart with an inverted pendulum.

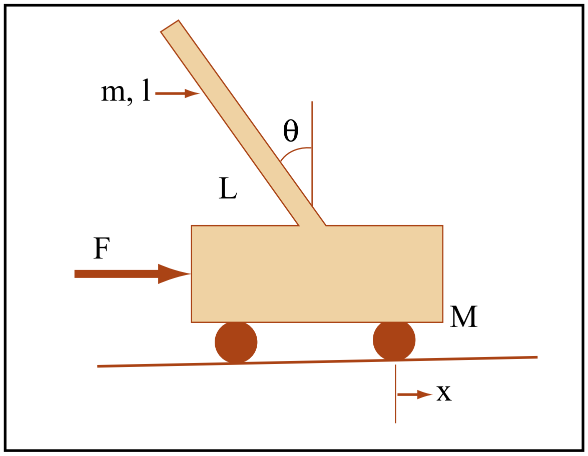


Figure by MIT OpenCourseWare.

- Nonlinear equations of motion can be developed for large angle motion (see 30-32)
- Force actuator, θ sensor

Linearize for small θ

$$(I+mL^2)\ddot{\theta} - mgL\,\theta = mL\ddot{x}$$

$$(M+m)\ddot{x} + g\dot{x} - mL\ddot{\theta} = F$$

$$\begin{bmatrix} (I+mL^2)s^2 - mgL & -mLs^2 \\ -mLs^2 & (M+m)s^2 + Gs \end{bmatrix}\begin{bmatrix} \Theta(s) \\ x(s) \end{bmatrix} = \begin{bmatrix} 0 \\ F(s) \end{bmatrix}$$

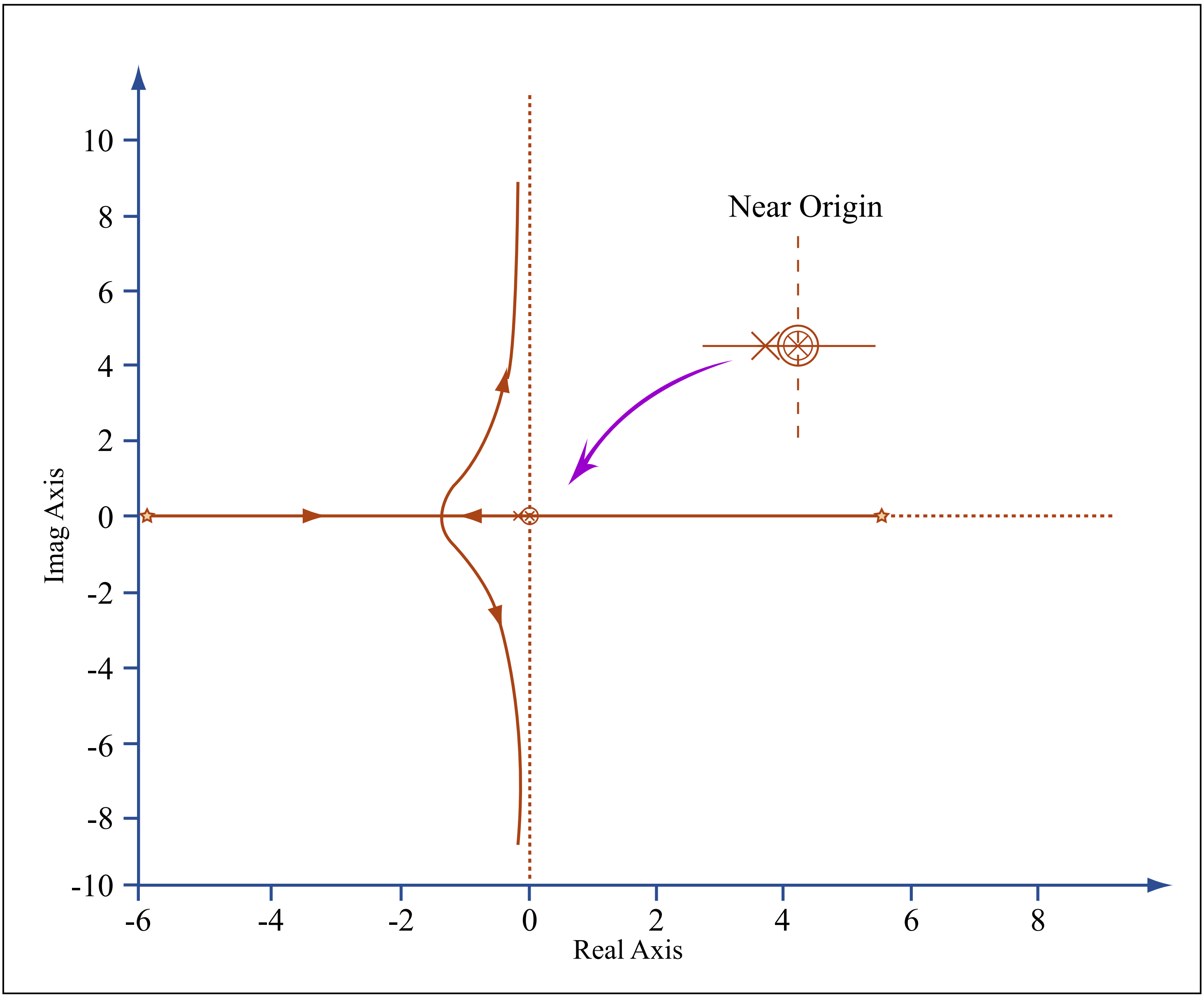
$$\frac{\Theta}{F} = \frac{mLs^2}{\left[(I+mL^2)s^2 - mgL\right]\left[(M+m)s^2 + Gs\right] - (mLs^2)^2}$$

Cannot say too much more

Let M= 0.5, m=0.2, G=0.1, I=0.006, L=0.3

➔ gives $$\frac{\Theta}{F} = \frac{4.54s^2}{s^4 + 0.1818s^3 - 31.18s^2 - 4.45s}$$

therefore has an unstable pole (as expected)
$s = \pm 5.6, -0.14, 0$



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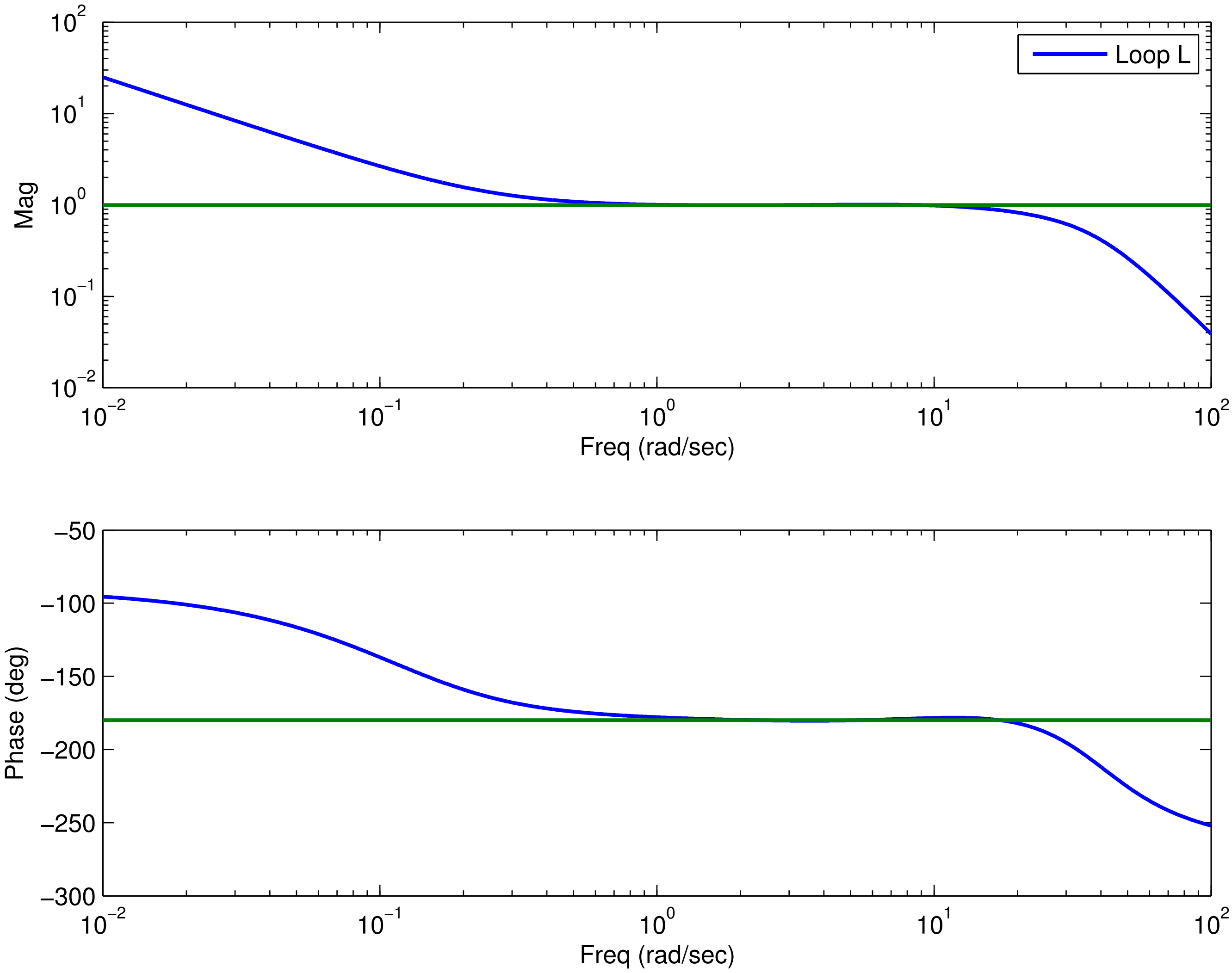


Figure 13.2: Loop and Margins

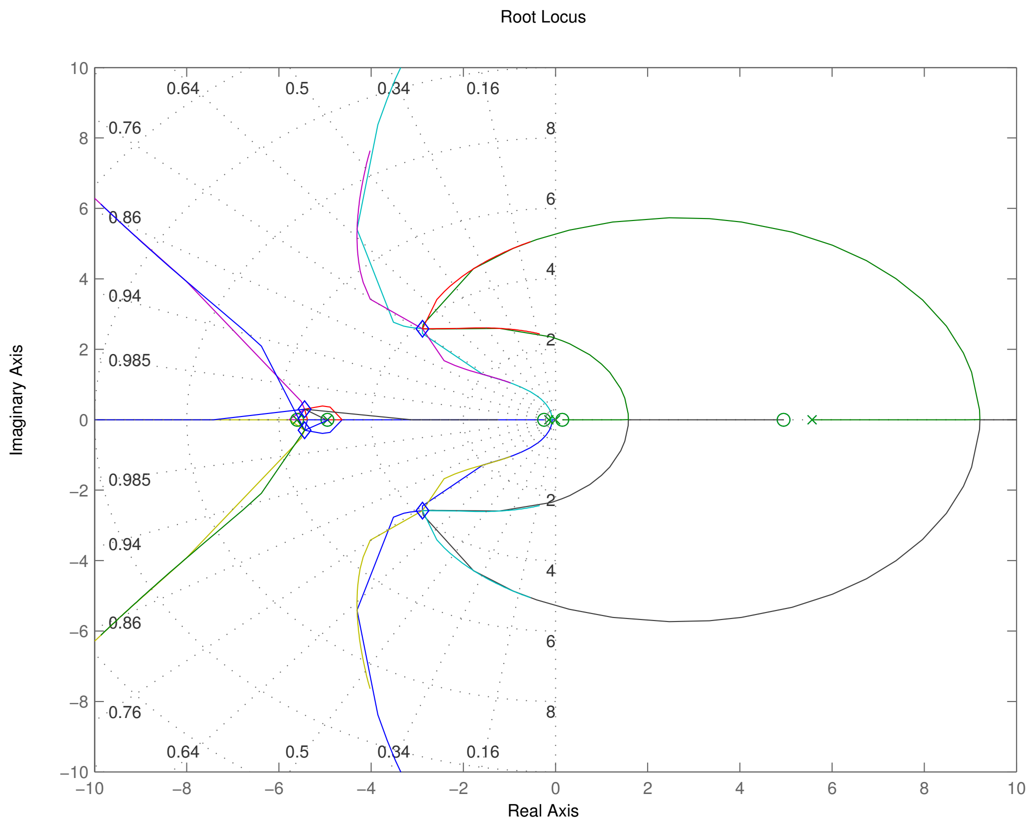
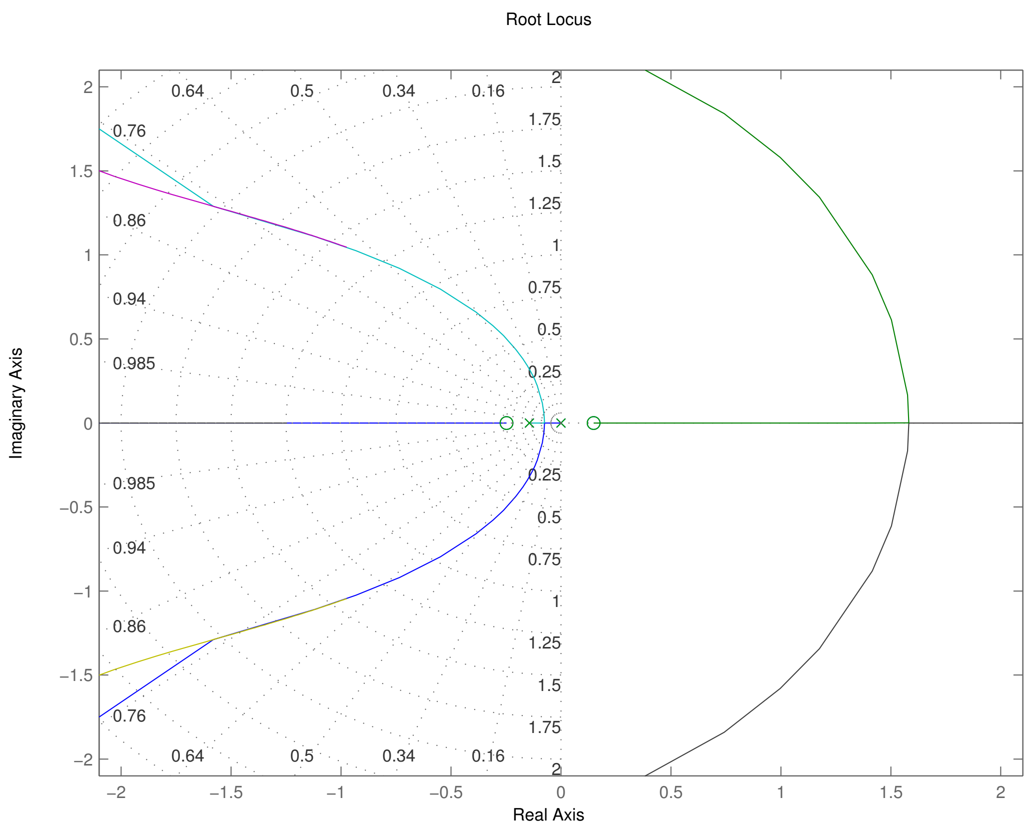


Figure 13.3: Root Locus with frozen compensator dynamics. Shows sensitivity to overall gain – symbols are a gain of [0.995:.0001:1.005].



- Looking at both the Loop TF plots and the root locus, it is clear this system is stable with a gain of 1, but
  - Unstable for a gain of $1 \pm \epsilon$ and/or a slight change in the system phase (possibly due to some unmodeled delays)
  - Very limited chance that this would work on the real system.

- Of course, this is an extreme example and not all systems are like this, but you must analyze to determine what **robustness margins** your controller really has.

- **Question:** what analysis tools should we use?

Analysis Tools to Use?

- Eigenvalues give a definite answer on the stability (or not) of the closed-loop system.
  - Problem is that it is very hard to predict where the closed-loop poles will go as a function of errors in the plant model.

- Consider the case were the model of the system is

$$\dot{x} = A_0 x + Bu$$

  - Controller also based on A_0, so **nominal** closed-loop dynamics:

$$\begin{bmatrix} A_0 & -BK \\ LC & A_0 - BK - LC \end{bmatrix} \Rightarrow \begin{bmatrix} A_0 - BK & BK \\ 0 & A_0 - LC \end{bmatrix}$$

- But what if the **actual** system has dynamics

$$\dot{x} = (A_0 + \Delta A)x + Bu$$

  - Then **perturbed** closed-loop system dynamics are:

$$\begin{bmatrix} A_0 + \Delta A & -BK \\ LC & A_0 - BK - LC \end{bmatrix} \Rightarrow \begin{bmatrix} A_0 + \Delta A - BK & BK \\ \Delta A & A_0 - LC \end{bmatrix}$$

- Transformed $\bar{A}_{cl}$ not upper-block triangular, so perturbed closed-loop eigenvalues are **NOT** the union of regulator & estimator poles.
  - Can find the closed-loop poles for a specific ΔA, but
  - Hard to predict change in location of closed-loop poles for a range of possible modeling errors.

Frequency Domain Tests

- Frequency domain stability tests provide further insights on the **stability margins**.

- Recall from the **Nyquist Stability Theorem:**
  - If the loop transfer function $L(s)$ has P poles in the RHP s-plane (and $\lim_{s\to\infty} L(s)$ is a constant), then for closed-loop stability, the locus of $L(\mathrm{j}\omega)$ for $\omega \in (-\infty, \infty)$ must encircle the critical point $(-1, 0)$ P times in the **counterclockwise** direction [Ogata 528].
  - This provides a binary measure of stability, or not.

- Can use "closeness" of $L(s)$ to the critical point as a measure of "closeness" to changing the number of encirclements.
  - Premise is that the system is stable for the nominal system $\Rightarrow$ has the right number of encirclements.

- Goal of the robustness test is to see if the possible perturbations to our system model (due to modeling errors) can **change the number of encirclements**
  - In this case, say that the perturbations can **destabilize** the system.

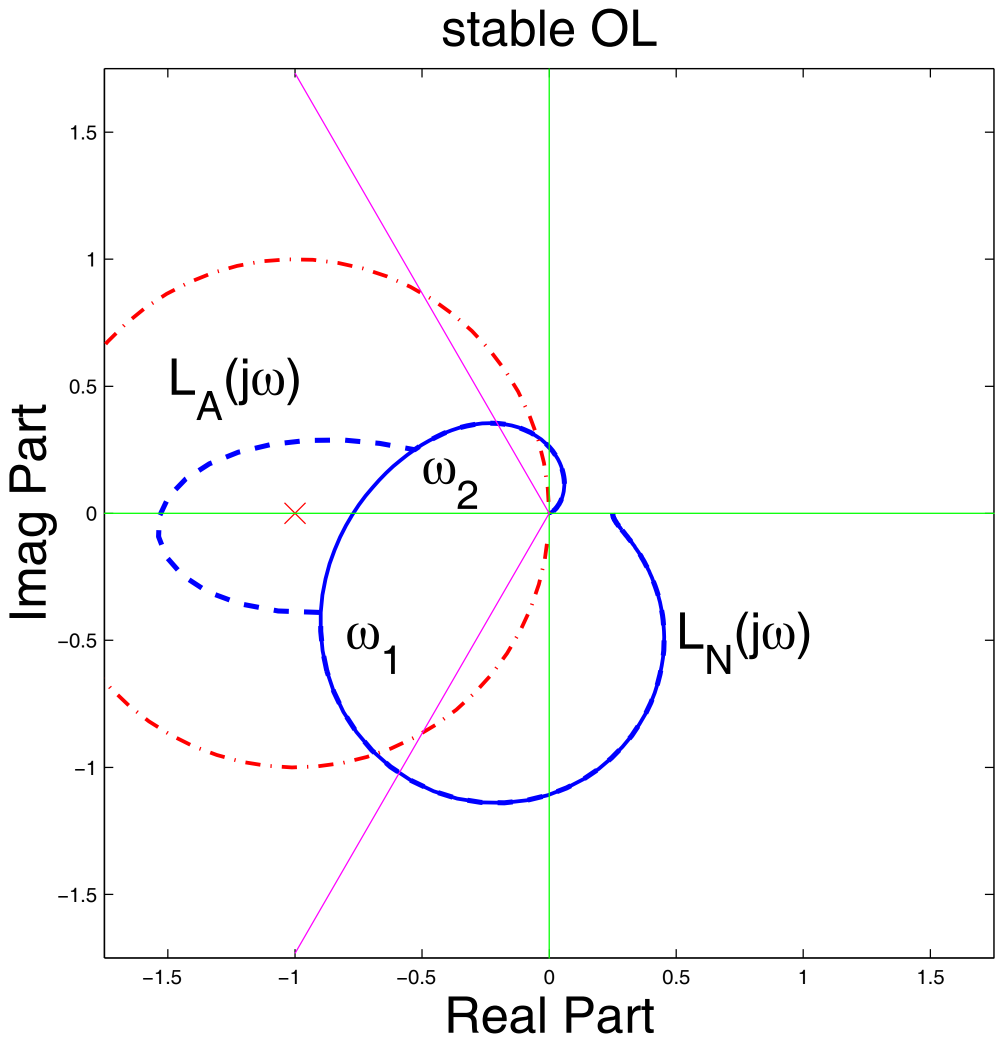
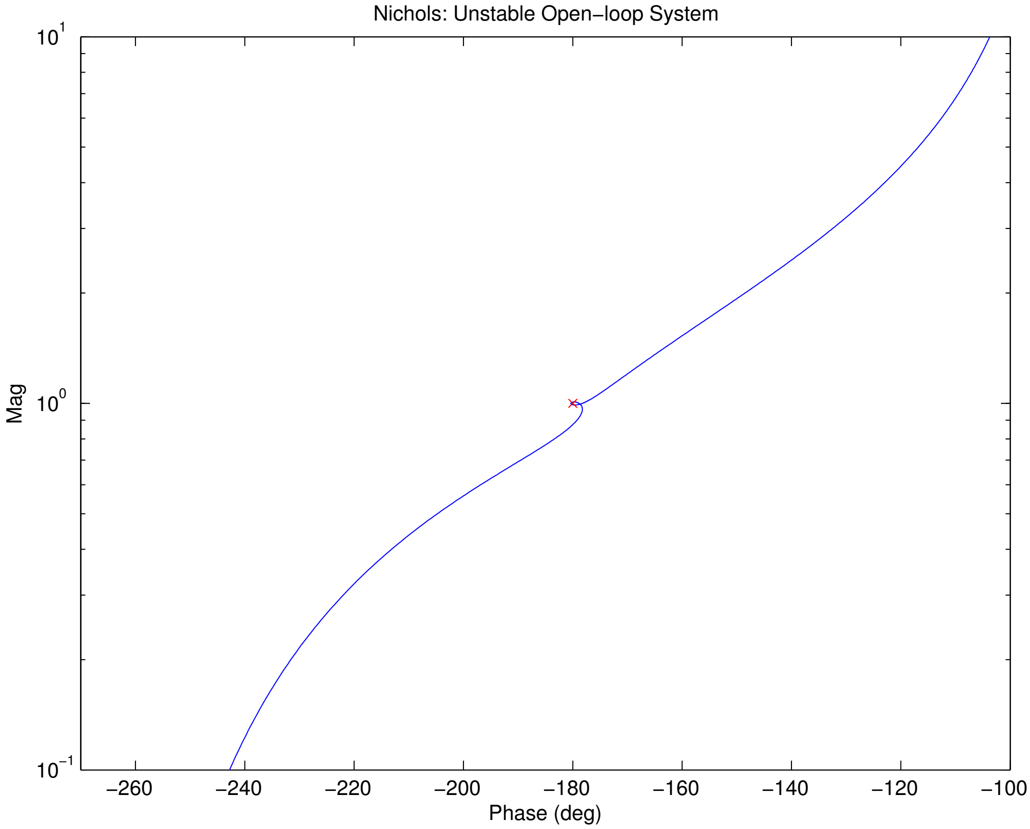


Figure 13.4: Plot of Loop TF $L_N(\mathrm{j}\omega) = G_N(\mathrm{j}\omega)G_c(\mathrm{j}\omega)$ and perturbation ($\omega_1 \rightarrow \omega_2$) that changes the number of encirclements.

- Model error in frequency range $\omega_1 \leq \omega \leq \omega_2$ causes a change in the number of encirclements of the critical point $(-1, 0)$
  - **Nominal** closed-loop system stable $L_N(s) = G_N(s)G_c(s)$
  - **Actual** closed-loop system unstable $L_A(s) = G_A(s)G_c(s)$

- **Bottom line:** Large model errors when $L_N \approx -1$ are very dangerous.



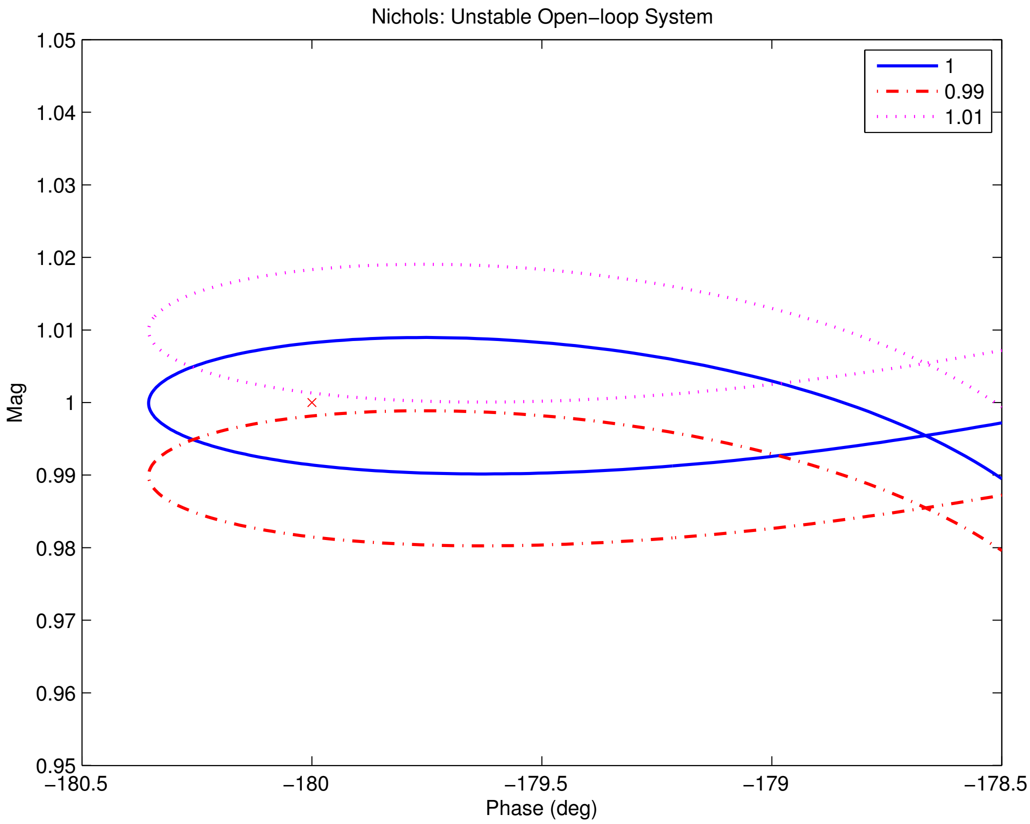


Figure 13.5: Nichols Plot ($|L((j\omega))|$ vs. $\arg L((j\omega))$) for the cart example which clearly shows the sensitivity to the overall gain and/or phase lag.

Frequency Domain Test

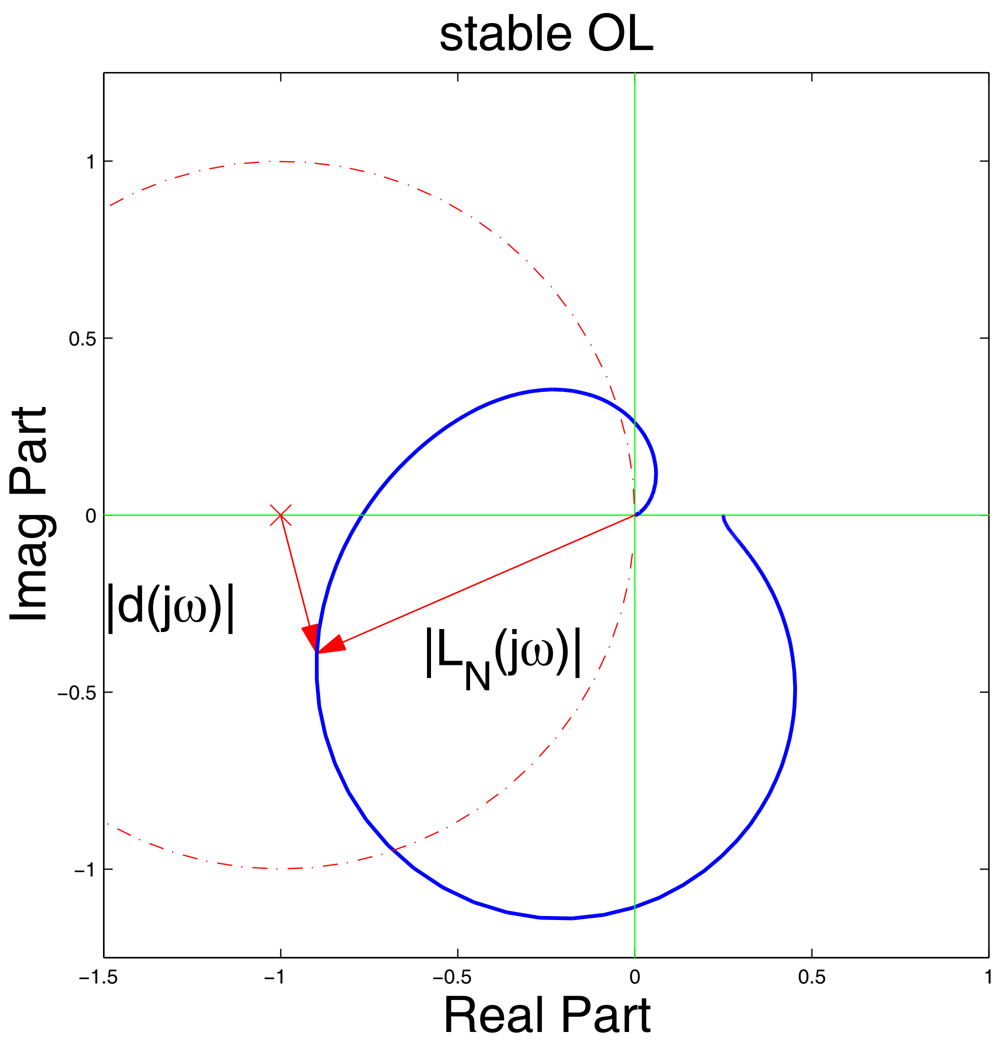


Figure 13.6: Geometric interpretation from Nyquist Plot of Loop TF.

- $|d(\mathrm{j}\omega)|$ measures distance of nominal Nyquist locus to critical point.

- But vector addition gives $$-1 + d(\mathrm{j}\omega) = L_N(\mathrm{j}\omega)$$

$$\Rightarrow \quad d(\mathrm{j}\omega) = 1 + L_N(\mathrm{j}\omega)$$

- Actually more convenient to plot
$$\frac{1}{|d(\mathrm{j}\omega)|} = \frac{1}{|1 + L_N(\mathrm{j}\omega)|} \triangleq |S(\mathrm{j}\omega)|$$
the magnitude of the **sensitivity transfer function** $S(s)$.

- So high sensitivity corresponds to $L_N(j\omega)$ being **very** close to the critical point.

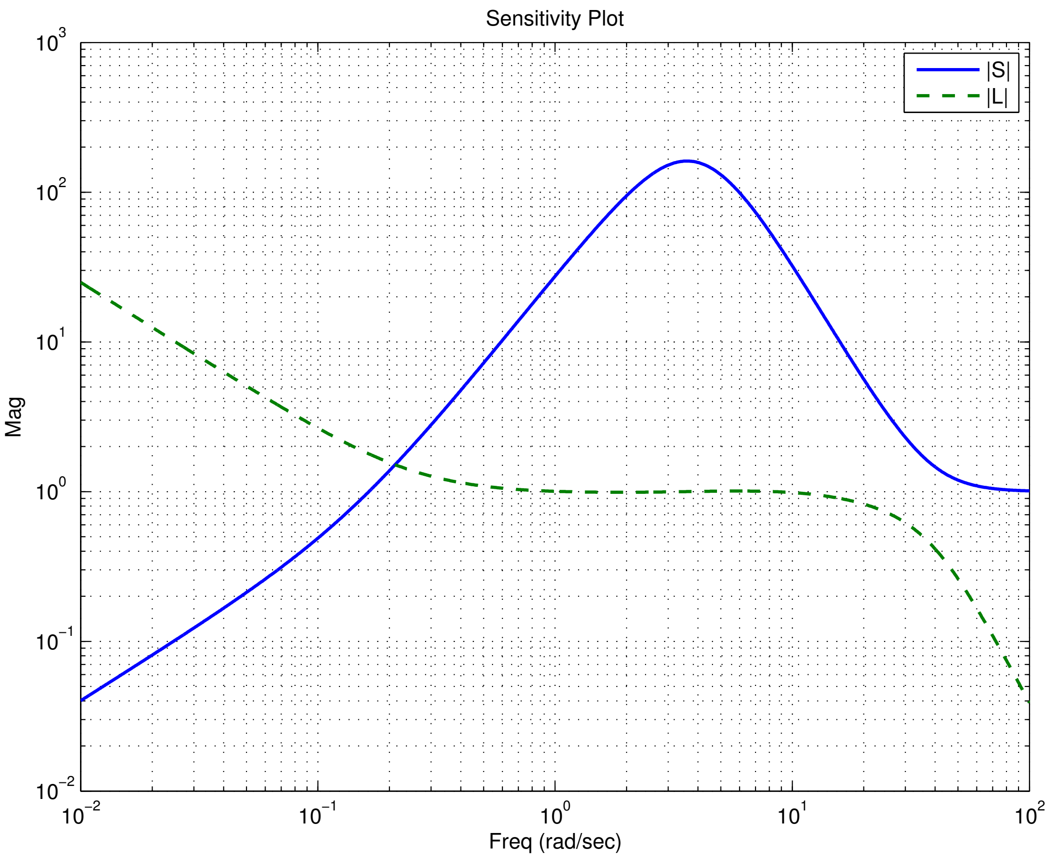


Figure 13.7: Sensitivity plot of the cart problem.

- Ideally you would want the sensitivity to be much lower than this.
  - Same as saying that you want $L(j\omega)$ to be far from the critical point.
  - Difficulty in this example is that the open-loop system is unstable, so $L(j\omega)$ must encircle the critical point $\Rightarrow$ hard for $L(j\omega)$ to get too far away from the critical point.

Summary

- LQG gives you a great way to design a controller for the nominal system.

- But there are no guarantees about the stability/performance if the actual system is slightly different.
  – Basic analysis tool is the **Sensitivity Plot**

- No obvious ways to tailor the specification of the LQG controller to improve any lack of robustness
  – Apart from the obvious "lower the controller bandwidth" approach.
  – And sometimes you need the bandwidth just to stabilize the system.

- Very hard to include additional robustness constraints into LQG
  – See my Ph.D. thesis in 1992.

- Other tools have been developed that allow you to **directly** shape the sensitivity plot $|S(j\omega)|$
  – Called $\mathcal{H}_\infty$ and μ

- **Good news:** Lack of robustness is something you should look for, but it is not always an issue.