16.323 Principles of Optimal Control
Spring 2008

16.323 Lecture 11

Estimators/Observers

- Bryson Chapter 12
- Gelb – Optimal Estimation

Estimators/Observers

- **Problem:** So far we have assumed that we have full access to the state $\mathbf{x}(t)$ when we designed our controllers.
  - Most often all of this information is not available.
  - And certainly there is usually error in our knowledge of $\mathbf{x}$.

- Usually can only feedback information that is developed from the sensors measurements.
  - Could try "output feedback" $\mathbf{u} = K\mathbf{x} \quad \Rightarrow \quad \mathbf{u} = \hat{K}\mathbf{y}$
  - But this is type of controller is hard to design.

- **Alternative approach:** Develop a replica of the dynamic system that provides an "estimate" of the system states based on the measured output of the system.

- **New plan:** called a "separation principle"
  1. Develop estimate of $\mathbf{x}(t)$, called $\hat{\mathbf{x}}(t)$.
  2. Then switch from $\mathbf{u} = -K\mathbf{x}(t)$ to $\mathbf{u} = -K\hat{\mathbf{x}}(t)$.

- Two key questions:
  - How do we find $\hat{\mathbf{x}}(t)$?
  - Will this new plan work? (yes, and very well)

Estimation Schemes

- Assume that the system model is of the form:

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}\ , \quad \mathbf{x}(0) \text{ unknown} \\ \mathbf{y} &= C_y\mathbf{x}\end{aligned}$$

  where
  - A, B, and C_y are known – possibly time-varying, but that is suppressed here.
  - $\mathbf{u}(t)$ is known
  - Measurable outputs are $\mathbf{y}(t)$ from $C_y \neq I$

- **Goal:** Develop a dynamic system whose state

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t) \quad \forall t \geq 0$$

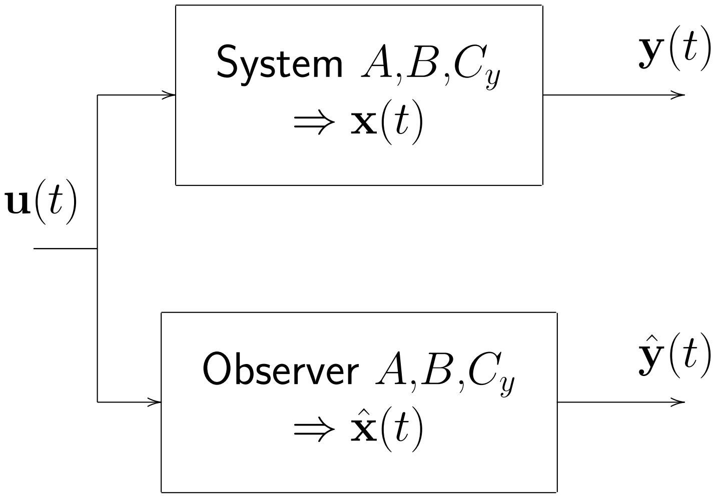
  Two primary approaches:
  - Open-loop.
  - Closed-loop.

Open-loop Estimator

- Given that we know the plant matrices and the inputs, we can just perform a simulation that runs in parallel with the system

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}} + B\mathbf{u}(t)$$

  – Then $\hat{\mathbf{x}}(t) \equiv \mathbf{x}(t)\ \forall\ t$ provided that $\hat{\mathbf{x}}(0) = \mathbf{x}(0)$



- To analyze this case, start with:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\end{aligned}$$

- Define the **estimation error**: $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$.
  – Now want $\tilde{\mathbf{x}}(t) = 0\ \forall\ t$, but is this realistic?

- **Major Problem:** We do not know $\mathbf{x}(0)$

- Subtract to get:

$$\frac{d}{dt}(\mathbf{x} - \hat{\mathbf{x}}) = A(\mathbf{x} - \hat{\mathbf{x}}) \quad \Rightarrow \quad \dot{\tilde{\mathbf{x}}}(t) = A\tilde{\mathbf{x}}$$

  which has the solution

$$\tilde{\mathbf{x}}(t) = e^{At}\tilde{\mathbf{x}}(0)$$

  – Gives the estimation error in terms of the initial error.

- Does this guarantee that $\tilde{\mathbf{x}} = 0 \;\forall\; t$?
  Or even that $\tilde{\mathbf{x}} \rightarrow 0$ as $t \rightarrow \infty$? (which is a more realistic goal).

  – Response is fine if $\tilde{\mathbf{x}}(0) = 0$. But what if $\tilde{\mathbf{x}}(0) \neq 0$?

- If A stable, then $\tilde{\mathbf{x}} \rightarrow 0$ as $t \rightarrow \infty$, but the dynamics of the estimation error are completely determined by the open-loop dynamics of the system (eigenvalues of A).
  – Could be very slow.
  – No obvious way to modify the estimation error dynamics.

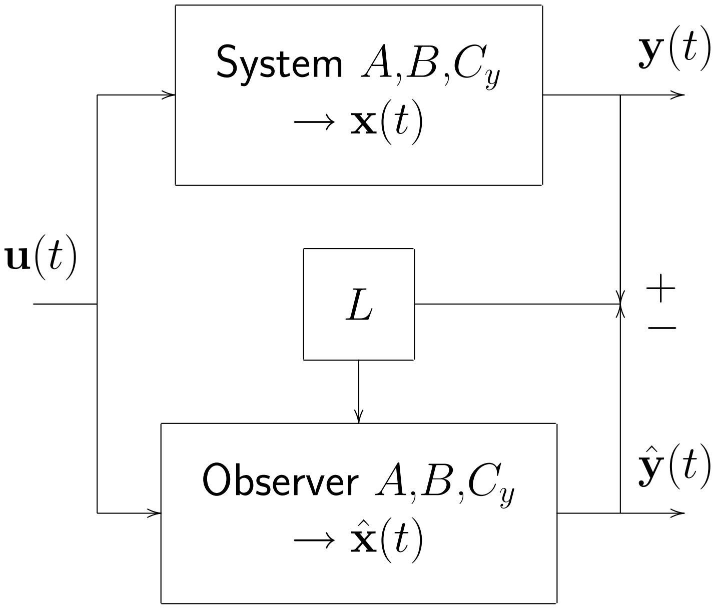
- Open-loop estimation **is not a very good idea.**

Closed-loop Estimator

- Obvious fix to problem: use the additional information available:
  – How well does the estimated output match the measured output?

$$\text{Compare: } \mathbf{y} = C_y\mathbf{x} \quad \text{with} \quad \hat{\mathbf{y}} = C_y\hat{\mathbf{x}}$$

  – Then form $\tilde{\mathbf{y}} = \mathbf{y} - \hat{\mathbf{y}} \equiv C_y\tilde{\mathbf{x}}$



- **Approach:** Feedback $\tilde{\mathbf{y}}$ to improve our estimate of the state. Basic form of the estimator is:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + \boxed{L\tilde{\mathbf{y}}(t)}\\ \hat{\mathbf{y}}(t) &= C_y\hat{\mathbf{x}}(t)\end{aligned}$$

  where L is a **user selectable gain matrix.**

- **Analysis:**

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} &= [A\mathbf{x} + B\mathbf{u}] - [A\hat{\mathbf{x}} + B\mathbf{u} + L(\mathbf{y} - \hat{\mathbf{y}})]\\ &= A(\mathbf{x} - \hat{\mathbf{x}}) - L(C\mathbf{x} - C_y\hat{\mathbf{x}})\\ &= A\tilde{\mathbf{x}} - LC_y\tilde{\mathbf{x}} = (A - LC_y)\tilde{\mathbf{x}}\end{aligned}$$

- So the closed-loop estimation error dynamics are now

$$\dot{\tilde{\mathbf{x}}} = (A - LC_y)\tilde{\mathbf{x}} \text{ with solution } \tilde{\mathbf{x}}(t) = e^{(A-LC_y)t}\,\tilde{\mathbf{x}}(0)$$

- **Bottom line:** Can select the gain L to attempt to improve the convergence of the estimation error (and/or speed it up).
  – But now must worry about observability of the system $[A, C_y]$.

- Note the similarity:
  – **Regulator Problem:** pick K for $A - BK$
    ◇ Choose $K \in \mathcal{R}^{1\times n}$ (SISO) such that the closed-loop poles

$$\det(sI - A + BK) = \Phi_c(s)$$

    are in the desired locations.
  – **Estimator Problem:** pick L for $A - LC_y$
    ◇ Choose $L \in \mathcal{R}^{n\times 1}$ (SISO) such that the closed-loop poles

$$\det(sI - A + LC_y) = \Phi_o(s)$$

    are in the desired locations.

- These problems are obviously very similar – in fact they are called **dual problems**
  – Note: poles of $(A - LC_y)$ and $(A - LC_y)^T$ are identical.
  – Also have that $(A - LC_y)^T = A^T - C_y^T L^T$
  – So designing L^T for this transposed system looks like a standard regulator problem $(A - BK)$ where

$$\begin{aligned} A &\Rightarrow A^T \\ B &\Rightarrow C_y^T \\ K &\Rightarrow L^T \end{aligned}$$

Estimator Example 10–1

- Simple system (see page 11-23)

$$A = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}$$
$$C_y = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

  – Assume that the initial conditions are not well known.
  – System stable, but $\lambda_{\max}(A) = -0.18$
  – Test observability:

$$\texttt{rank}\begin{bmatrix} C_y \\ C_y A \end{bmatrix} = \texttt{rank}\begin{bmatrix} 1 & 0 \\ -1 & 1.5 \end{bmatrix}$$

- Use open and closed-loop estimators. Since the initial conditions are not well known, use $\hat{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- Open-loop estimator:

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= A\hat{\mathbf{x}} + B\mathbf{u} \\ \hat{\mathbf{y}} &= C_y\hat{\mathbf{x}} \end{aligned}$$

- Closed-loop estimator:

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= A\hat{\mathbf{x}} + B\mathbf{u} + L\tilde{\mathbf{y}} = A\hat{\mathbf{x}} + B\mathbf{u} + L(\mathbf{y} - \hat{\mathbf{y}}) \\ &= (A - LC_y)\hat{\mathbf{x}} + B\mathbf{u} + L\mathbf{y} \\ \hat{\mathbf{y}} &= C_y\hat{\mathbf{x}} \end{aligned}$$

  – Dynamic system with poles $\lambda_i(A - LC_y)$ that takes the measured plant outputs as an input and generates an estimate of $\mathbf{x}$.
  – Use `place` command to set closed-loop pole locations

- Typically simulate both systems together for simplicity

- Open-loop case:

$$
\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\
\mathbf{y} &= C_y\mathbf{x} \\
\dot{\hat{\mathbf{x}}} &= A\hat{\mathbf{x}} + B\mathbf{u} \\
\hat{\mathbf{y}} &= C_y\hat{\mathbf{x}}
\end{aligned}
$$

$$
\Rightarrow \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} \mathbf{u} \ , \quad \begin{bmatrix} \mathbf{x}(0) \\ \hat{\mathbf{x}}(0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} \mathbf{y} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} C_y & 0 \\ 0 & C_y \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix}
$$

- Closed-loop case:

$$
\begin{aligned}
\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\
\dot{\hat{\mathbf{x}}} &= (A - LC_y)\hat{\mathbf{x}} + B\mathbf{u} + LC_y\mathbf{x}
\end{aligned}
$$

$$
\Rightarrow \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\hat{\mathbf{x}}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC_y & A - LC_y \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} \mathbf{u}
$$

- Example uses a strong $\mathbf{u}(t)$ to shake things up

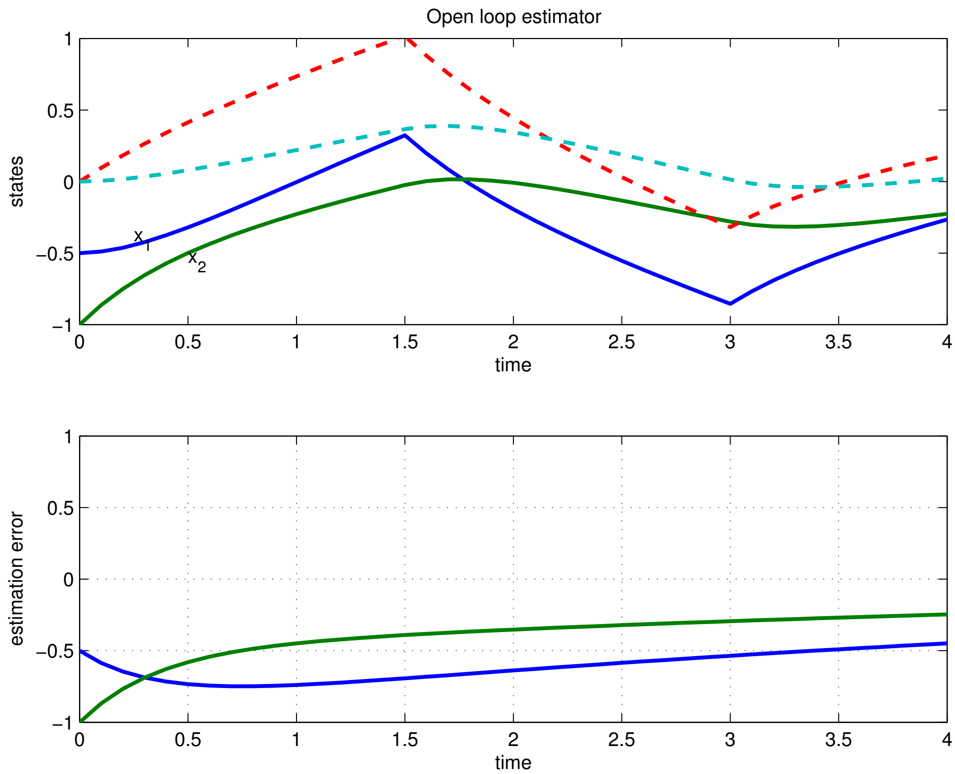


Figure 11.1: Open-loop estimator. Estimation error converges to zero, but very slowly.

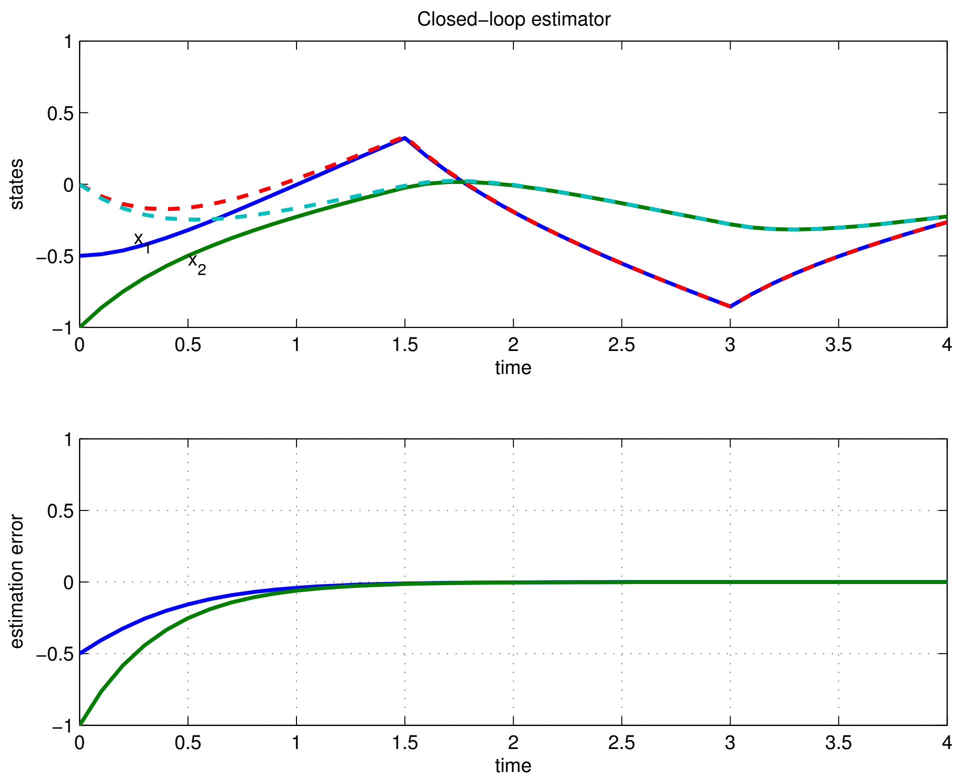


Figure 11.2: Closed-loop estimator. Convergence looks much better.

Estimator Poles?

- Location heuristics for poles still apply – use Bessel, ITAE, ...
  - Main difference: probably want to make the estimator faster than you intend to make the regulator – should enhance the control, which is based on $\hat{\mathbf{x}}(t)$.
  - ROT: Factor of 2–3 in the time constant $\zeta\omega_n$ associated with the regulator poles.

- **Note:** When designing a regulator, were concerned with "bandwidth" of the control getting too high $\Rightarrow$ often results in control commands that *saturate* the actuators and/or change rapidly.

- Different concerns for the estimator:
  - Loop closed inside computer, so saturation not a problem.
  - However, the measurements $\mathbf{y}$ are often "noisy", and we need to be careful how we use them to develop our state estimates.

$\Rightarrow$ **High bandwidth estimators** tend to accentuate the effect of sensing noise in the estimate.
  - State estimates tend to "track" the measurements, which are fluctuating randomly due to the noise.

$\Rightarrow$ **Low bandwidth estimators** have lower gains and tend to rely more heavily on the plant model
  - Essentially an open-loop estimator – tends to ignore the measurements and just uses the plant model.

- Can also develop an **optimal estimator** for this type of system.
  - Given duality of regulator and estimator, would expect to see close connection between optimal estimator and regulator (LQR)

- Key step is to **balance** the effect of the various types of random noise in the system on the estimator:

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + B_w\mathbf{w} \\ \mathbf{y} &= C_y\mathbf{x} + \mathbf{v}\end{aligned}$$

  - $\mathbf{w}$: "process noise" – models uncertainty in the system model.
  - $\mathbf{v}$: "sensor noise" – models uncertainty in the measurements.

- Typically assume that $\mathbf{w}(t)$ and $\mathbf{v}(t)$ are zero mean $E[\mathbf{w}(t)] = 0$ and
  - Uncorrelated Gaussian white random noises: no correlation between the noise at one time instant and another

$$\begin{aligned}E[\mathbf{w}(t_1)\mathbf{w}(t_2)^T] &= R_{\text{ww}}(t_1)\delta(t_1 - t_2) \quad \Rightarrow \mathbf{w}(t) \sim \mathcal{N}(0, R_{\text{ww}}) \\ E[\mathbf{v}(t_1)\mathbf{v}(t_2)^T] &= R_{\text{vv}}(t_1)\delta(t_1 - t_2) \quad \Rightarrow \mathbf{v}(t) \sim \mathcal{N}(0, R_{\text{vv}}) \\ E[\mathbf{w}(t_1)\mathbf{v}(t_2)^T] &= 0\end{aligned}$$

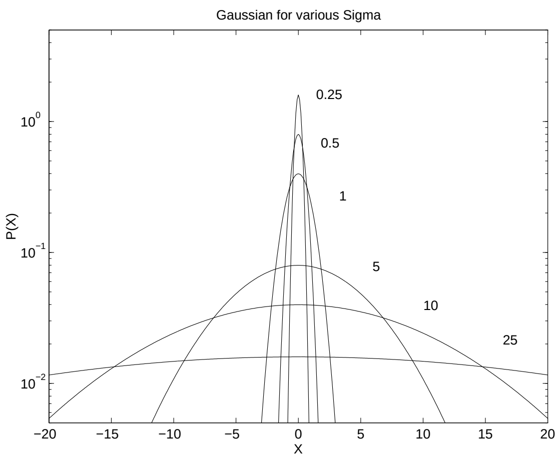


Figure 11.3: Example of impact of covariance $= \sigma^2$ on the distribution of the PDF – wide distribution corresponds to large uncertainty in the variable

Analysis

- With noise in the system, the model is of the form:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + B_w\mathbf{w}\ , \quad \mathbf{y} = C_y\mathbf{x} + \mathbf{v}$$

  – And the estimator is of the form:

$$\dot{\hat{\mathbf{x}}} = A\hat{\mathbf{x}} + B\mathbf{u} + L(\mathbf{y} - \hat{\mathbf{y}})\ , \quad \hat{\mathbf{y}} = C_y\hat{\mathbf{x}}$$

- **Analysis:** in this case:

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}} &= \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = [A\mathbf{x} + B\mathbf{u} + B_w\mathbf{w}] - [A\hat{\mathbf{x}} + B\mathbf{u} + L(\mathbf{y} - \hat{\mathbf{y}})] \\
&= A(\mathbf{x} - \hat{\mathbf{x}}) - L(C_y\mathbf{x} - C_y\hat{\mathbf{x}}) + B_w\mathbf{w} - L\mathbf{v} \\
&= A\tilde{\mathbf{x}} - LC_y\tilde{\mathbf{x}} + B_w\mathbf{w} - L\mathbf{v} \\
&= (A - LC_y)\tilde{\mathbf{x}} + B_w\mathbf{w} - L\mathbf{v} \qquad (11.18)
\end{aligned}$$

- This equation of the estimation error explicitly shows the **conflict** in the estimator design process. Must **balance** between:
  – Speed of the estimator decay rate, which is governed by

$$\texttt{Re}[\lambda_i(A - LC_y)]$$

  – Impact of the sensing noise $\mathbf{v}$ through the gain L

- Fast state reconstruction requires rapid decay rate – typically requires a large L, but that tends to magnify the effect of $\mathbf{v}$ on the estimation process.
  – The effect of the process noise is always there, but the choice of L will tend to mitigate/accentuate the effect of $\mathbf{v}$ on $\tilde{\mathbf{x}}(t)$.

- **Kalman Filter** needs to provide an optimal balance between the two conflicting problems for a given "size" of the process and sensing noises.

- Note that Eq. 11.18 is of the form of a linear time-varying system driven by white Gaussian noise
  – Can predict the **mean square value** of the state (estimation error in this case) $Q(t) = E[\tilde{\mathbf{x}}(t)\tilde{\mathbf{x}}(t)^T]$ over time using $Q(0) = Q_0$ and

$$\begin{aligned}\dot{Q}(t) &= [A - LC_y]\,Q(t) + Q(t)\,[A - LC_y]^T \\ &\qquad + \begin{bmatrix} B_w & -L \end{bmatrix} \begin{bmatrix} R_{\rm ww} & 0 \\ 0 & R_{\rm vv} \end{bmatrix} \begin{bmatrix} B_w^T \\ -L^T \end{bmatrix} \\ &= [A - LC_y]\,Q(t) + Q(t)\,[A - LC_y]^T + B_w R_{\rm ww} B_w^T + L R_{\rm vv} L^T\end{aligned}$$

  – Called a matrix **differential Lyapunov Equation**[16]

- Note that ideally would like to minimize $Q(t)$ or `trace` $Q(t)$, but that is difficult to do & describe easily[17].

- Instead, consider option of trying to minimize `trace` $\dot{Q}(t)$, the argument being that then $\int_0^t$ `trace` $\dot{Q}(\tau)d\tau$ is small.
  – Not quite right, but good enough to develop some insights

- To proceed note that

$$\frac{\partial}{\partial X}\texttt{trace}[AXB] = \frac{\partial}{\partial X}\texttt{trace}[B^T X^T A^T] = A^T B^T$$

  and

$$\frac{\partial}{\partial X}\texttt{trace}[AXBX^TC] = A^T C^T X B^T + CAXB$$

- So for minimum we require that

$$\frac{\partial}{\partial L}\texttt{trace}\ \dot{Q} = -2Q^T C_y^T + 2LR_{\rm vv} = 0$$

  which implies that

$$L = Q(t) C_y^T R_{\rm vv}^{-1}$$

[16]See K+S, chapter 1.11 for details.

[17]My 16.324 discuss how to pose the problem in discrete time and then let $\Delta t \to 0$ to recover the continuous time results.

- Note that if we use this expression for L in the original differential Lyapunov Equation, we obtain:

$$\begin{aligned}
\dot{Q}(t) &= [A - LC_y]\, Q(t) + Q(t)\, [A - LC_y]^T + B_w R_{\rm ww} B_w^T + L R_{\rm vv} L^T \\
&= \left[A - Q(t)C_y^T R_{\rm vv}^{-1} C_y\right] Q(t) + Q(t) \left[A - Q(t)C_y^T R_{\rm vv}^{-1} C_y\right]^T \\
&\qquad + B_w R_{\rm ww} B_w^T + Q(t)C_y^T R_{\rm vv}^{-1} R_{\rm vv} (Q(t)C_y^T R_{\rm vv}^{-1})^T \\
&= AQ(t) + Q(t)A^T - 2Q(t)C_y^T R_{\rm vv}^{-1} C_y Q(t) + B_w R_{\rm ww} B_w^T \\
&\qquad + Q(t)C_y^T R_{\rm vv}^{-1} C_y Q(t) \\
\dot{Q}(t) &= AQ(t) + Q(t)A^T + B_w R_{\rm ww} B_w^T - Q(t)C_y^T R_{\rm vv}^{-1} C_y Q(t)
\end{aligned}$$

which is obviously a matrix differential Riccati equation.

Optimal Kalman Filter

- **Goal:** develop an estimator $\hat{\mathbf{x}}(t)$ which is a linear function of the measurements $\mathbf{y}(\tau)$ $(0 \leq \tau \leq t)$ and **minimizes** the function

$$J = \texttt{trace}(Q(t))$$
$$Q(t) = E\left[\{\mathbf{x}(t) - \hat{\mathbf{x}}(t)\}\{\mathbf{x}(t) - \hat{\mathbf{x}}(t)\}^T\right]$$

which is the **covariance** for the **estimation error**.

- **Solution:** is a closed-loop estimator [18]

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}} + L(t)(\mathbf{y}(t) - C_y\hat{\mathbf{x}}(t))$$

where $L(t) = Q(t)C_y^T R_{\rm vv}^{-1}$ and $Q(t) \geq 0$ solves

$$\dot{Q}(t) = AQ(t) + Q(t)A^T + B_w R_{\rm ww} B_w^T - Q(t)C_y^T R_{\rm vv}^{-1} C_y Q(t)$$

  - Note that $\hat{x}(0)$ and $Q(0)$ are known
  - Differential equation for $Q(t)$ **solved forward in time**.
  - **Filter form** of the **differential matrix Riccati** equation for the **error covariance**.
  - Note that the $AQ(t) + Q(t)A^T \ldots$ is different than with the regulator which had $P(t)A + A^T P(t) \ldots$

- Called **Kalman-Bucy Filter** – **linear quadratic estimator (LQE)**

[18] See OCW notes for 16.322 "Stochastic Estimation and Control" for the details of this derivation.

- Note that an increase in $Q(t)$ corresponds to **increased uncertainty in the state estimate**. $\dot{Q}(t)$ has several contributions:
  - $AQ(t) + Q(t)A^T$ is the homogeneous part
  - $B_w R_{\text{ww}} B_w^T$ increase due to the process measurements
  - $Q(t)C_y^T R_{\text{vv}}^{-1} C_y Q(t)$ decrease due to measurements

- The estimator gain is $L(t) = Q(t)C_y^T R_{\text{vv}}^{-1}$
  - Feedback on the **innovation**, $\mathbf{y} - \hat{\mathbf{y}}$
  - If the uncertainty about the state is high, then $Q(t)$ is large, and so the innovation $\mathbf{y} - C_y\hat{\mathbf{x}}$ is weighted heavily ($L \uparrow$)
  - If the measurements are very accurate $R_{\text{vv}} \downarrow$, then the measurements are heavily weighted

Steady State

- Assume that [19]
  1. $R_{\text{vv}} > 0$, $R_{\text{ww}} > 0$
  2. All plant dynamics are constant in time
  3. $[A, C_y]$ detectable
  4. $[A, B_w]$ stabilizable

- Then, as with the LQR problem, the covariance of the LQE quickly settles down to a constant Q_{ss} independent of $Q(0)$, as $t \to \infty$ where

$$AQ_{ss} + Q_{ss}A^T + B_w R_{\text{ww}} B_w^T - Q_{ss} C_y^T R_{\text{vv}}^{-1} C_y Q_{ss} = 0$$

  – Stabilizable/detectable gives a unique $Q_{ss} \geq 0$
  – $Q_{ss} > 0$ iff $[A, B_w]$ controllable
  – $L_{ss} = Q_{ss} C_y^T R_{\text{vv}}^{-1}$

- If Q_{ss} exists, the steady state filter

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}} + L_{ss}(\mathbf{y}(t) - C_y \hat{\mathbf{x}}(t)) \\ &= (A - L_{ss} C_y)\hat{\mathbf{x}}(t) + L_{ss}\mathbf{y}(t) \end{aligned}$$

  is asymptotically stable iff (1)–(4) above hold.

[19] Compare this with 4–10

Filter Interpretation

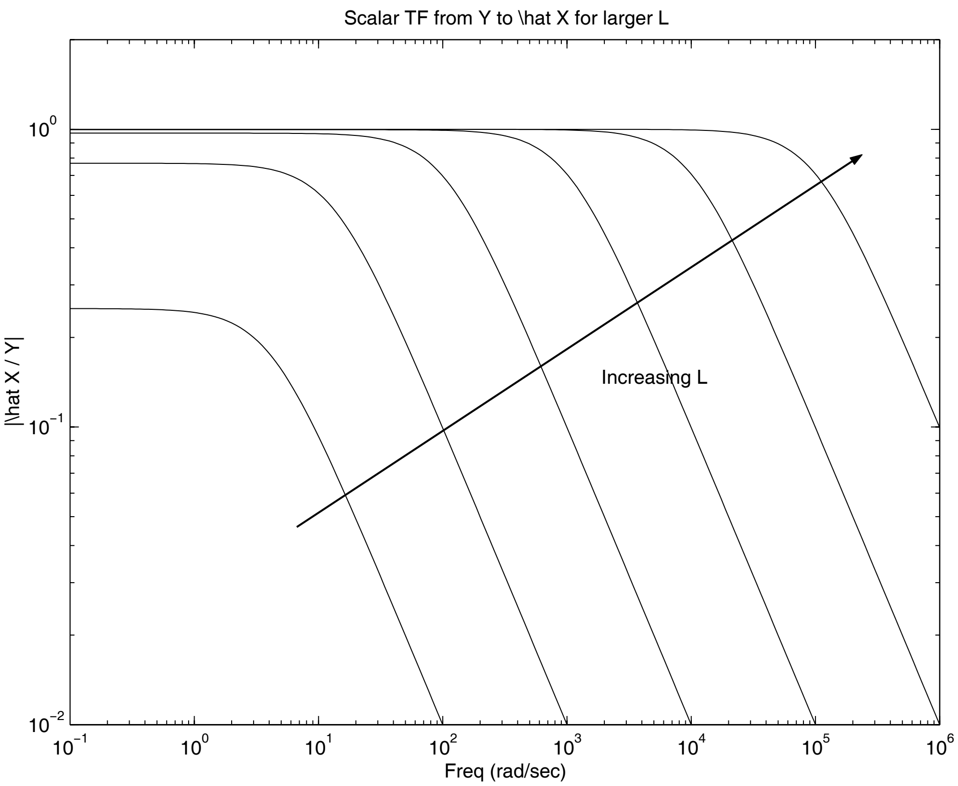
- Given that $\dot{\hat{\mathbf{x}}} = (A - LC_y)\hat{\mathbf{x}} + L\mathbf{y}$

- Consider a scalar system, and take the Laplace transform of both sides to get:

$$\frac{\hat{X}(s)}{Y(s)} = \frac{L}{sI - (A - LC_y)}$$

- This is the transfer function from the "measurement" to the "estimated state"
  - It looks like a low-pass filter.

- Clearly, by lowering $R_{\rm vv}$, and thus increasing L, we are pushing out the pole.
  - DC gain asymptotes to $1/C_y$ as $L \to \infty$



Example 10–2

- Lightly Damped Harmonic Oscillator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

  and $y = x_1 + v$, where $R_{\text{ww}} = 1$ and $R_{\text{vv}} = r$.

  – Can sense the position state of the oscillator, but want to develop an estimator to reconstruct the velocity state.

- **Symmetric root locus** exists for the optimal estimator. Can find location of the optimal poles using a SRL based on the TF

$$G_{yw}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \omega_0^2 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{s^2 + \omega_0^2} = \frac{N(s)}{D(s)}$$

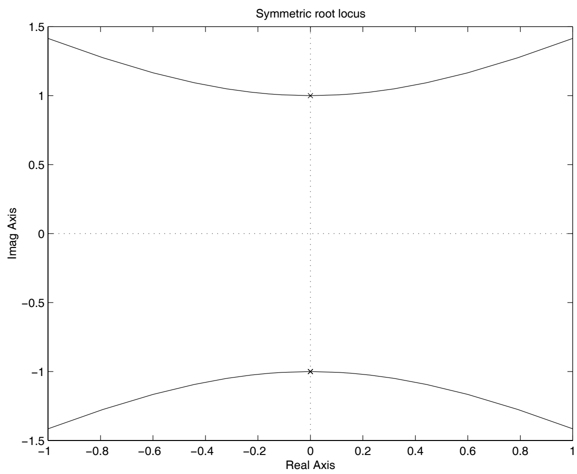
  – SRL for the closed-loop poles $\lambda_i(A - LC)$ of the estimator which are the LHP roots of:

$$D(s)D(-s) \pm \frac{R_{\text{ww}}}{R_{\text{vv}}} N(s)N(-s) = 0$$

  – Pick sign to ensure that there are no poles on the $\mathbf{j}\omega$-axis (other than for a gain of zero)

  – So we must find the LHP roots of

$$\left[s^2 + \omega_0^2\right] \left[(-s)^2 + \omega_0^2\right] + \frac{1}{r} = (s^2 + \omega_0^2)^2 + \frac{1}{r} = 0$$



- Note that as $r \rightarrow 0$ (clean sensor), the estimator poles tend to ∞ along the $\pm$45 deg asymptotes, so the poles are approximately

$$s \approx \frac{-1 \pm j}{\sqrt{r}} \quad \Rightarrow \quad \Phi_e(s) = s^2 + \frac{2}{\sqrt{r}}s + \frac{2}{r} = 0$$

- Can use these estimate pole locations in `acker`, to get that

$$\begin{aligned} L &= \left(\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}^2 + \frac{2}{\sqrt{r}} \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} + \frac{2}{r} I \right) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{r} - \omega_0^2 & \frac{2}{\sqrt{r}} \\ -\frac{2}{\sqrt{r}}\omega_0^2 & \frac{2}{r} - \omega_0^2 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{r}} \\ \frac{2}{r} - \omega_0^2 \end{bmatrix} \end{aligned}$$

- Given L, A, and C, we can develop the estimator transfer function from the measurement y to the $\hat{x}_2$

$$\begin{aligned} \frac{\hat{x}_2}{y} &= \begin{bmatrix} 0 & 1 \end{bmatrix} \left(sI - \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{\sqrt{r}} \\ \frac{2}{r} - \omega_0^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{2}{\sqrt{r}} \\ \frac{2}{r} - \omega_0^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + \frac{2}{\sqrt{r}} & -1 \\ \frac{2}{r} & s \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{\sqrt{r}} \\ \frac{2}{r} - \omega_0^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 1 \\ \frac{-2}{r} & s + \frac{2}{\sqrt{r}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{r}} \\ \frac{2}{r} - \omega_0^2 \end{bmatrix} \frac{1}{s^2 + \frac{2}{\sqrt{r}}s + \frac{2}{r}} \\ &= \frac{\frac{-2}{r}\frac{2}{\sqrt{r}} + (s + \frac{2}{\sqrt{r}})(\frac{2}{r} - \omega_0^2)}{s^2 + \frac{2}{\sqrt{r}}s + \frac{2}{r}} \approx \frac{s - \sqrt{r}\omega_0^2}{s^2 + \frac{2}{\sqrt{r}}s + \frac{2}{r}} \end{aligned}$$

- Filter zero asymptotes to $s = 0$ as $r \rightarrow 0$ and the two poles $\rightarrow \infty$
- Resulting estimator looks like a "band-limited" differentiator.
  - Expected because we measure position and want to estimate velocity.
  - Frequency band over which filter performs differentiation determined by the "relative cleanliness" of the measurements.

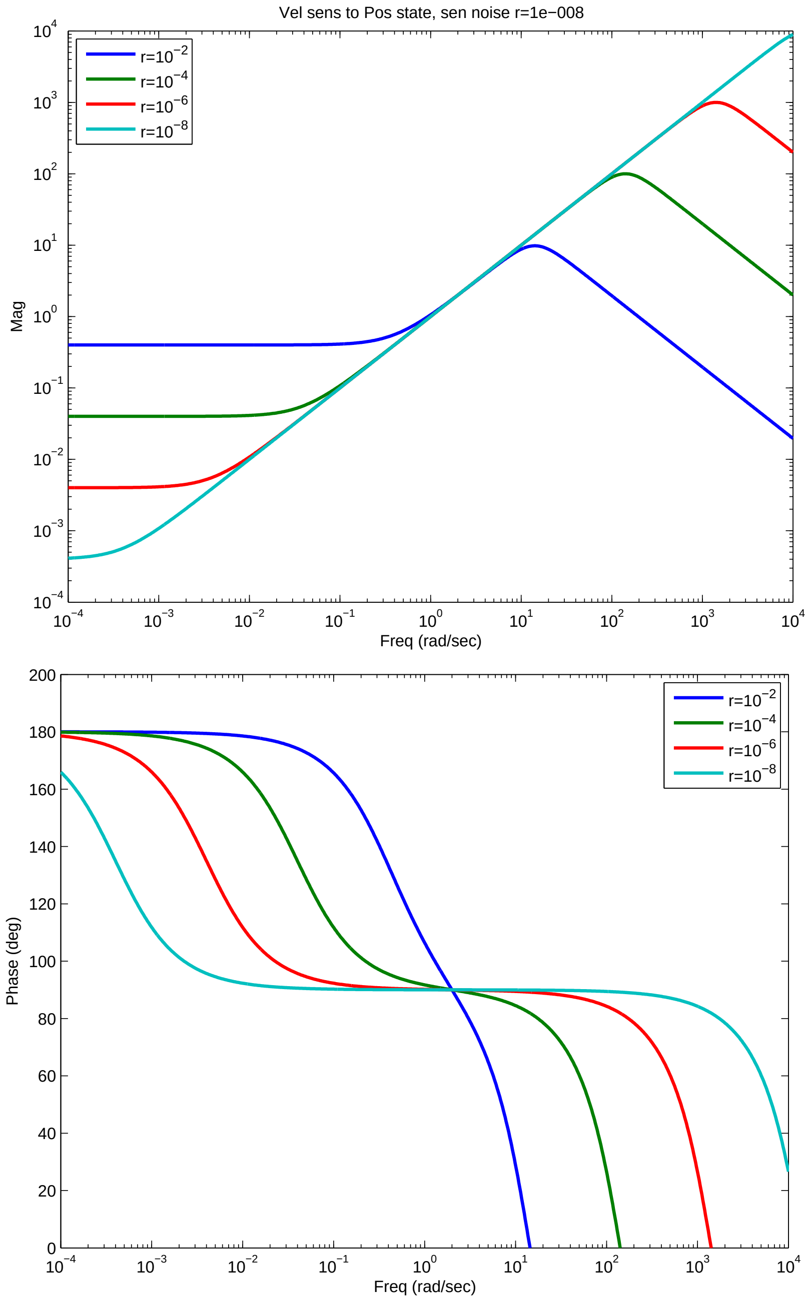


Figure 11.4: Bandlimited differentiation of the position measurement from LQE: $r = 10^{-2}$, $r = 10^{-4}$, $r = 10^{-6}$, and $r = 10^{-8}$

Final Thoughts

- Note that the feedback gain L in the estimator only stabilizes the estimation error.
  - If the system is unstable, then the state estimates will also go to ∞, with zero error from the actual states.

- Estimation is an important concept of its own.
  - Not always just "part of the control system"
  - Critical issue for guidance and navigation system

- More complete discussion requires that we study stochastic processes and optimization theory.

- **Estimation is all about which do you trust more: your measurements or your model.**

- Strong duality between LQR and LQE problems

$$\begin{array}{ccc} A & \leftrightarrow & A^T \\ B & \leftrightarrow & C_y^T \\ C_z & \leftrightarrow & B_w^T \\ R_{\rm zz} & \leftrightarrow & R_{\rm ww} \\ R_{\rm uu} & \leftrightarrow & R_{\rm vv} \\ K(t) & \leftrightarrow & L^T(t_f - t) \\ P(t) & \leftrightarrow & Q(t_f - t) \end{array}$$

Basic Estimator (examp1.m) (See page 11-7)

```
% Examples of estimator performance
% Jonathan How, MIT
% 16.333 Fall 2005
%
% plant dynamics
%
a=[-1 1.5;1 -2];b=[1 0]';c=[1 0];d=0;
%
% estimator gain calc
%
l=place(a',c',[-3 -4]);l=l'
%
% plant initial cond
xo=[-.5;-1];
% extimator initial cond
xe=[0 0]';
t=[0:.1:10];
%
% inputs
%
u=0;u=[ones(15,1);-ones(15,1);ones(15,1)/2;-ones(15,1)/2;zeros(41,1)];
%
% open-loop extimator
%
A_ol=[a zeros(size(a));zeros(size(a)) a];
B_ol=[b;b];
C_ol=[c zeros(size(c));zeros(size(c)) c];
D_ol=zeros(2,1);
%
% closed-loop extimator
%
A_cl=[a zeros(size(a));l*c a-l*c];B_cl=[b;b];
C_cl=[c zeros(size(c));zeros(size(c)) c];D_cl=zeros(2,1);

[y_cl,x_cl]=lsim(A_cl,B_cl,C_cl,D_cl,u,t,[xo;xe]);
[y_ol,x_ol]=lsim(A_ol,B_ol,C_ol,D_ol,u,t,[xo;xe]);

figure(1);clf;subplot(211)
plot(t,x_cl(:,[1 2]),t,x_cl(:,[3 4]),'--','LineWidth',2);axis([0 4 -1 1]);
title('Closed-loop estimator');ylabel('states');xlabel('time')
text(.25,-.4,'x_1');text(.5,-.55,'x_2');subplot(212)
plot(t,x_cl(:,[1 2])-x_cl(:,[3 4]),'LineWidth',2)
%setlines;
axis([0 4 -1 1]);grid on
ylabel('estimation error');xlabel('time')

figure(2);clf;subplot(211)
plot(t,x_ol(:,[1 2]),t,x_ol(:,[3 4]),'--','LineWidth',2);axis([0 4 -1 1])
title('Open loop estimator');ylabel('states');xlabel('time')
text(.25,-.4,'x_1');text(.5,-.55,'x_2');subplot(212)
plot(t,x_ol(:,[1 2])-x_ol(:,[3 4]),'LineWidth',2)
%setlines;
axis([0 4 -1 1]);grid on
ylabel('estimation error');xlabel('time')

print -depsc -f1 est11.eps; jpdf('est11')
print -depsc -f2 est12.eps; jpdf('est12')
```

Filter Interpretation

```
% Simple LQE example showing SRL
% 16.323 Spring 2007
% Jonathan How
%
a=[0 1;-4 0];
c=[1 0]; % pos sensor
c2=[0 1]; % vel state out
f=logspace(-4,4,800);

r=1e-2;
l=polyvalm([1 2/sqrt(r) 2/r],a)*inv([c;c*a])*[0 1]'
[nn,dd]=ss2tf(a-l*c,l,c2,0); % to the vel estimate
g=freqresp(nn,dd,f*j);
[r roots(nn)]
figure(1)
subplot(211)
f1=f;g1=g;
loglog(f,abs(g))
%hold on;fill([5e2 5e2 1e3 1e3 5e2]',[1e4 1e-4 1e-4 1e4 1e4]','c');hold off
xlabel('Freq (rad/sec)')
ylabel('Mag')
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
axis([1e-3 1e3 1e-4 1e4])
subplot(212)
semilogx(f,unwrap(angle(g))*180/pi)
xlabel('Freq (rad/sec)')
ylabel('Phase (deg)')
axis([1e-3 1e3 0 200])

figure(2)
r=1e-4;
l=polyvalm([1 2/sqrt(r) 2/r],a)*inv([c;c*a])*[0 1]'
[nn,dd]=ss2tf(a-l*c,l,c2,0); % to the vel estimate
g=freqresp(nn,dd,f*j);
[r roots(nn)]
subplot(211)
f2=f;g2=g;
loglog(f,abs(g))
%hold on;fill([5e2 5e2 1e3 1e3 5e2]',[1e4 1e-4 1e-4 1e4 1e4]','c');hold off
xlabel('Freq (rad/sec)')
ylabel('Mag')
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
axis([1e-3 1e3 1e-4 1e4])
subplot(212)
semilogx(f,unwrap(angle(g))*180/pi)
xlabel('Freq (rad/sec)')
ylabel('Phase (deg)')
%bode(nn,dd);
axis([1e-3 1e3 0 200])

figure(3)
r=1e-6;
l=polyvalm([1 2/sqrt(r) 2/r],a)*inv([c;c*a])*[0 1]'
[nn,dd]=ss2tf(a-l*c,l,c2,0); % to the vel estimate
g=freqresp(nn,dd,f*j);
[r roots(nn)]
subplot(211)
f3=f;g3=g;
loglog(f,abs(g))
%hold on;fill([5e2 5e2 1e3 1e3 5e2]',[1e4 1e-4 1e-4 1e4 1e4]','c');hold off
xlabel('Freq (rad/sec)')
ylabel('Mag')
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
axis([1e-3 1e3 1e-4 1e4])
subplot(212)
semilogx(f,unwrap(angle(g))*180/pi)
xlabel('Freq (rad/sec)')
```

```
ylabel('Phase (deg)')
%bode(nn,dd);
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
axis([1e-3 1e3 0 200])

figure(4)
r=1e-8;
l=polyvalm([1 2/sqrt(r) 2/r],a)*inv([c;c*a])*[0 1]'
[nn,dd]=ss2tf(a-l*c,l,c2,0); % to the vel estimate
g=freqresp(nn,dd,f*j);
[r roots(nn)]
f4=f;g4=g;
subplot(211)
loglog(f,abs(g))
%hold on;fill([5e2 5e2 1e3 1e3 5e2]',[1e4 1e-4 1e-4 1e4 1e4]','c');hold off
xlabel('Freq (rad/sec)')
ylabel('Mag')
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
axis([1e-3 1e3 1e-4 1e4])
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
subplot(212)
semilogx(f,unwrap(angle(g))*180/pi)
xlabel('Freq (rad/sec)')
ylabel('Phase (deg)')
%bode(nn,dd);
axis([1e-3 1e3 0 200])

print -depsc -f1 filt1.eps; jpdf('filt1')
print -depsc -f2 filt2.eps;jpdf('filt2')
print -depsc -f3 filt3.eps;jpdf('filt3')
print -depsc -f4 filt4.eps;jpdf('filt4')

figure(5);clf
%subplot(211)
loglog(f1,abs(g1),f2,abs(g2),f3,abs(g3),f4,abs(g4),'Linewidth',2)
%hold on;fill([5e2 5e2 1e3 1e3 5e2]',[1e4 1e-4 1e-4 1e4 1e4]','c');hold off
xlabel('Freq (rad/sec)')
ylabel('Mag')
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
axis([1e-4 1e4 1e-4 1e4])
title(['Vel sens to Pos state, sen noise r=',num2str(r)])
legend('r=10^{-2}','r=10^{-4}','r=10^{-6}','r=10^{-8}','Location','NorthWest')
%subplot(212)
figure(6);clf
semilogx(f1,unwrap(angle(g1))*180/pi,f2,unwrap(angle(g2))*180/pi,...
    f3,unwrap(angle(g3))*180/pi,f4,unwrap(angle(g4))*180/pi,'Linewidth',2);hold off
xlabel('Freq (rad/sec)')
ylabel('Phase (deg)')
legend('r=10^{-2}','r=10^{-4}','r=10^{-6}','r=10^{-8}')
%bode(nn,dd);
axis([1e-4 1e4 0 200])
print -depsc -f5 filt5.eps;jpdf('filt5')
print -depsc -f6 filt6.eps;jpdf('filt6')
```