16.323 Principles of Optimal Control
Spring 2008

16.323 Lecture 10

Singular Arcs

- Bryson Chapter 8
- Kirk Section 5.6

Singular Problems

- There are occasions when the PMP

$$\mathbf{u}^\star(t) = \arg\left\{\min_{\mathbf{u}(t)\in\mathcal{U}} H(\mathbf{x},\mathbf{u},\mathbf{p},t)\right\}$$

fails to define $\mathbf{u}^\star(t)$ $\Rightarrow$ can an extremal control still exist?

– Typically occurs when the Hamiltonian is linear in the control, and the coefficient of the **control term equals zero**.

- Example: on page 9-10 we wrote the control law:

$$u(t) = \begin{cases} -u_m & b < p_2(t) \\ 0 & -b < p_2(t) < b \\ u_m & p_2(t) < -b \end{cases}$$

but we do not know what happens if $p_2 = b$ for an interval of time.

– Called a **singular arc**.

– Bottom line is that the straightforward solution approach does not work, and we need to investigate the PMP conditions in more detail.

- **Key point**: depending on the system and the cost, singular arcs might exist, and we must determine their existence to fully characterize the set of possible control solutions.

- Note: control on the singular arc is determined by the requirements that the coefficient of the linear control terms in $H_{\mathbf{u}}$ remain zero on the singular arc and so must the time derivatives of $H_{\mathbf{u}}$.

– Necessary condition for scalar u can be stated as

$$(-1)^k \frac{\partial}{\partial u}\left[\left(\frac{d^{2k}}{dt^{2k}}\right) H_u\right] \geq 0 \quad k = 0, 1, 2 \ldots$$

- With $\dot{x} = u$, $x(0) = 1$ and $0 \leq u(t) \leq 4$, consider objective

$$\min \int_0^2 (x(t) - t^2)^2 dt$$

- First form standard Hamiltonian

$$H = (x(t) - t^2)^2 + p(t)u(t)$$

which gives $H_u = p(t)$ and

$$\dot{p}(t) = -H_x = -2(x - t^2), \qquad \text{with } p(2) = 0 \tag{10.15}$$

- Note that if $p(t) > 0$, then PMP indicates that we should take the minimum possible value of $u(t) = 0$.
  – Similarly, if $p(t) < 0$, we should take $u(t) = 4$.

- Question: can we get that $H_u \equiv 0$ for some interval of time?
  – Note: $H_u \equiv 0$ implies $p(t) \equiv 0$, which means $\dot{p}(t) \equiv 0$, and thus

$$\dot{p}(t) \equiv 0 \Rightarrow x(t) = t^2, \quad u(t) = \dot{x} = 2t$$

- Thus we get the control law that

$$u(t) = \begin{cases} 0 & p(t) > 0 \\ 2t \quad \text{when} & p(t) = 0 \\ 4 & p(t) < 0 \end{cases}$$

- Can show by contradiction that optimal solution has $x(t) \geq t^2$ for $t \in [0, 2]$.
  - And thus we know that $\dot{p}(t) \leq 0$ for $t \in [0, 2]$
  - But $p(2) = 0$ and $\dot{p}(t) \leq 0$ imply that $p(t) \geq 0$ for $t \in [0, 2]$

- So there must be a point in time $k \in [0, 2]$ after which $p(t) = 0$ (some steps skipped here...)
  - Check options: $k = 0$? $\Rightarrow$ contradiction
  - Check options: $k = 2$? $\Rightarrow$ contradiction

- So must have $0 < k < 2$. How find it? Control law will be

$$u(t) = \begin{cases} 0 & \text{when } 0 \leq t < k \\ 2t & \qquad\; k \leq t < 2 \end{cases}$$

  apply this control to the state equations and get:

$$x(t) = \begin{cases} 1 & \text{when } 0 \leq t \leq k \\ t^2 + (1 - k^2) & \qquad\; k \leq t \leq 2 \end{cases}$$

  To find k, note that must have $p(t) \equiv 0$ for $t \in [k, 2]$, so in this time range

$$\dot{p}(t) \equiv 0 = -2(1 - k^2) \quad \Rightarrow \quad k = 1$$

  - So now both $u(t)$ and $x(t)$ are known, and the optimal solution is to "bang off" and then follow a singular arc.

Singular Arc Example 2

- LTI system, $x_1(0)$, $x_2(0)$, t_f given; $x_1(t_f) = x_2(t_f) = 0$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $J = \frac{1}{2}\int_0^{t_f} x_1^2 dt$ (see Bryson and Ho, p. 248)

- So $H = \frac{1}{2}x_1(t)^2 + p_1(t)x_2(t) + p_1(t)u(t) - p_2(t)u(t)$

$$\Rightarrow \quad \dot{p}_1(t) = -x_1(t), \qquad \dot{p}_2(t) = -p_1(t)$$

- For a singular arc, we must have $H_u = 0$ for a finite time interval

$$H_u = p_1(t) - p_2(t) = 0?$$

- Thus, during that interval

$$\begin{aligned} \frac{d}{dt}H_u &= \dot{p}_1(t) - \dot{p}_2(t) \\ &= -x_1(t) + p_1(t) = 0 \end{aligned}$$

- Note that H is not an explicit function of time, so H is a **constant for all time**

$$H = \frac{1}{2}x_1(t)^2 + p_1(t)x_2(t) + [p_1(t) - p_2(t)]\, u(t) = C$$

but can now substitute from above along the singular arc

$$\frac{1}{2}x_1(t)^2 + x_1(t)x_2(t) = C$$

which gives a family of singular arcs in the state x_1, x_2

- To find the appropriate control to stay on the arc, use

$$\frac{d^2}{dt^2}(H_u) = -\dot{x}_1 + \dot{p}_1 = -(x_2(t) + u(t)) - x_1(t) = 0$$

or that $u(t) = -(x_1(t) + x_2(t))$ which is a linear feedback law to use along the singular arc.

Details for LTI Systems

- Consider the min time-fuel problem for the general system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

with $M^- \leq u_i \leq M^+$ and

$$J = \int_0^{t_f} (1 + \sum_{i=1}^{m} c_i |u_i|) dt$$

t_f is free and we want to drive the state to the origin

- We studied this case before, and showed that

$$H = 1 + \sum_{i=1}^{m} (c_i |u_i| + \mathbf{p}^T B_i u_i) + \mathbf{p}^T A\mathbf{x}$$

- On a singular arc, $\frac{d^k}{dt^k}(H_u) = 0 \Rightarrow$ coefficient of u in H is zero

$$\Rightarrow \mathbf{p}^T(t) B_i = \pm c_i$$

for non-zero period of time and

$$\frac{d^k}{dt^k}(\mathbf{p}^T(t) B_i) = \left(\frac{d^k \mathbf{p}(t)}{dt^k}\right)^T B_i = 0 \ \ \forall\, k \geq 1$$

- Recall the necessary conditions $\dot{\mathbf{p}}^T = -H_{\mathbf{x}} = -\mathbf{p}^T A$, which imply

$$\begin{aligned}
\ddot{\mathbf{p}}^T &= -\dot{\mathbf{p}}^T A = \mathbf{p}^T A^2 \\
\dddot{\mathbf{p}}^T &= -\ddot{\mathbf{p}}^T A = -\mathbf{p}^T A^3 \\
&\vdots \\
\left(\frac{d^k \mathbf{p}(t)}{dt^k}\right)^T &\equiv (-1)^k \mathbf{p}^T A^k
\end{aligned}$$

and combining with the above gives

$$\left(\frac{d^k \mathbf{p}(t)}{dt^k}\right)^T B_i = (-1)^k \mathbf{p}^T A^k B_i = 0$$

- Rewriting these equations yields the conditions that

$$\mathbf{p}^T A B_i = 0, \qquad \mathbf{p}^T A^2 B_i = 0, \qquad \cdots$$

$$\Rightarrow \mathbf{p}^T A \left[\begin{array}{cccc} B_i & AB_i & \cdots & A^{n-1}B_i \end{array} \right] = 0$$

- There are three ways to get:

$$\mathbf{p}^T A \left[\begin{array}{cccc} B_i & AB_i & \cdots & A^{n-1}B_i \end{array} \right] = 0$$

- On a singular arc, we know that $\mathbf{p}(t) \neq 0$ so this does not cause the condition to be zero.

- What if A singular, and $\mathbf{p}(t)^T A = 0$ on the arc?
  - Then $\dot{\mathbf{p}}^T = -\mathbf{p}^T A = 0$. In this case, $\mathbf{p}(t)$ is constant over $[t_0, t_f]$
  - Indicates that if the problem is singular at any time, it is singular for all time.
  - This also indicates that $\mathbf{u}$ is a constant.
  - A possible case, but would be unusual since it is very restrictive set of control inputs.

- Third possibility is that $\left[\begin{array}{cccc} B_i & AB_i & \cdots & A^{n-1}B_i \end{array} \right]$ is singular, meaning that the system is not controllable by the individual control inputs.
  - Very likely scenario – most common cause of singularity conditions.
  - Lack of controllability by a control input does not necessarily mean that a singular arc has to exist, but it is a possibility.

- For **Min Time** problems, now $c_i = 0$, so things are a bit different

- In this case the switchings are at $\mathbf{p}^T B_i = 0$ and a similar analysis as before gives the condition that

$$\mathbf{p}^T \left[\begin{array}{cccc} B_i & AB_i & \cdots & A^{n-1}B_i \end{array} \right] = 0$$

- Now there are only 2 possibilities
  - $\mathbf{p} = 0$ is one, but in that case,

$$H = 1 + \mathbf{p}^T(A\mathbf{x} + B\mathbf{u}) = 1$$

    but we would expect that $H = 0$
  - Second condition is obviously the lack of controllability again.

- **Summary (Min time)**:
  - If the system is completely controllable by B_i, then u_i can have no singular intervals
  - Not shown, but if the system is not completely controllable by B_i, then u_i **must** have a singular interval.

- **Summary (Min time-fuel)**:
  - If the system is completely controllable by B_i and A is non-singular, then there can be no singular intervals

Nonlinear Systems

- Consider systems that are nonlinear in the state, but linear in the control

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t)) + \mathbf{b}(\mathbf{x}(t))\mathbf{u}(t)$$

with cost

$$J = \int_{t_0}^{t_f} \mathbf{g}(\mathbf{x}(t))dt$$

- For a singular arc, in general you will find that

$$\frac{d^k}{dt^k}\left(H_{u_i}\right) = 0 \qquad k = 0, \ldots, r-1$$

but these conditions provide no indication of the control required to keep the system on the singular arc
– i.e. the coefficient of the control terms is zero.

- But then for some r and i, $\frac{d^r}{dt^r}\left(H_{u_i}\right) = 0$ does retain u_i.
– So if $\mathbf{u}_j(\mathbf{x}, \mathbf{p})$ are the other control inputs, then

$$\frac{d^r}{dt^r}\left(H_{u_i}\right) = C(\mathbf{x}, \mathbf{p}, \mathbf{u}_j(\mathbf{x}, \mathbf{p})) + D(\mathbf{x}, \mathbf{p}, \mathbf{u}_j(\mathbf{x}, \mathbf{p}))u_i = 0$$

with $D \neq 0$, so the condition does depend on u_i.

- Then can define the appropriate control law to stay on the singular arc as

$$u_i = -\frac{C(\mathbf{x}, \mathbf{p}, \mathbf{u}_j(\mathbf{x}, \mathbf{p},))}{D(\mathbf{x}, \mathbf{p}, \mathbf{u}_j(\mathbf{x}, \mathbf{p},))}$$

- Properties of this solution are:
  - $r \geq 2$ is even
  - Singular surface of dimension $2n - r$ in space of $(\mathbf{x}, \mathbf{p})$ in general, but $2n - r - 1$ if t_f is free (additional constraint that $H(t) = 0$)
  - Additional necessary condition for the singular arc to be extremal is that:

$$(-1)^{r/2} \frac{\partial}{\partial u_i} \left[\frac{d^r}{dt^r} H_u \right] \geq 0$$

  - Note that in the example above,

$$\frac{\partial}{\partial u_i} \left[\frac{d^r}{dt^r} H_u \right] \sim D$$

Last Example

- Goddard problem: thrust program for maximum altitude of a sounding rocket [Bryson and Ho, p. 253]. Given the EOM:

$$\begin{aligned} \dot{v} &= \frac{1}{m}[F(t) - D(v,h)] - g \\ \dot{h} &= v \\ \dot{m} &= \frac{-F(t)}{c} \end{aligned}$$

where g is a constant, and drag model is $D(v,h) = \frac{1}{2}\rho v^2 C_d S e^{-\beta h}$

- **Problem**: Find $0 \leq F(t) \leq F_{\max}$ to maximize $h(t_f)$ with $v(0) = h(0) = 0$ and $m(0), m(t_f)$ are given

- The Hamiltonian is

$$H = p_1 \left(\frac{1}{m}[F(t) - D(v,h)] - g \right) + p_2 v - p_3 \frac{F(t)}{c}$$

and since $v(t_f)$ is not specified and we are maximizing $h(t_f)$,

$$p_2(t_f) = -1 \qquad p_1(t_f) = 0$$

  – Note that $H(t) = 0$ since the final time is not specified.

- The costate EOM are:

$$\dot{\mathbf{p}} = \begin{bmatrix} \frac{1}{m}\frac{\partial D}{\partial v} & -1 & 0 \\ \frac{1}{m}\frac{\partial D}{\partial h} & 0 & 0 \\ \frac{F-D}{m^2} & 0 & 0 \end{bmatrix} \mathbf{p}$$

- H is linear in the controls, and the minimum is found by minimizing $(\frac{p_1}{m} - \frac{p_3}{c})F(t)$, which clearly has 3 possible solutions:

$$\begin{array}{ccc} F = F_{\max} & & (\frac{p_1}{m} - \frac{p_3}{c}) < 0 \\ 0 < F < F_{\max} & \text{if} & (\frac{p_1}{m} - \frac{p_3}{c}) = 0 \\ F = 0 & & (\frac{p_1}{m} - \frac{p_3}{c}) > 0 \end{array}$$

  – Middle expression corresponds to a singular arc.

- Note: on a singular arc, must have $H_u = p_1 c - p_3 m = 0$ for finite interval, so then $\dot{H}_u = 0$ and $\ddot{H}_u = 0$, which means

$$\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) p_1 - m p_2 = 0$$

and

$$F = D + mg + \frac{m}{D + 2c\frac{\partial D}{\partial v} + c^2 \frac{\partial^2 D}{\partial v^2}} \cdot \left[-g(D + c\frac{\partial D}{\partial v}) + c(c - v)\frac{\partial D}{\partial h} - vc^2 \frac{\partial^2 D}{\partial v \partial h}\right] \quad (10.16)$$

which is a nonlinear feedback control law for thrust on a singular arc.
  - For this particular drag model, the feedback law simplifies to:

$$F = D + mg + \frac{mg}{1 + 4(c/v) + 2(c/v)^2} \left[\frac{\beta c^2}{g}\left(1 + \frac{v}{c}\right) - 1 - 2\frac{c}{v}\right]$$

  and the singular surface is: $mg = \left(1 + \frac{v}{c}\right) D$

- Constraints $H(t) = 0$, $H_u = 0$, and $\dot{H}_u = 0$ provide a condition that defines a surface for the singular arc in v, h, m space:

$$D + mg - \frac{v}{c} D - v \frac{\partial D}{\partial v} = 0 \quad (10.17)$$

- It can then be shown that the solution typically consists of 3 arcs:
  1. $F = F_{\max}$ until 10.17 is satisfied.
  2. Follow singular arc using 10.16 feedback law until $m(t) = m(t_f)$.
  3. $F = 0$ until $v = 0$.

  which is of the form "bang-singular-bang"

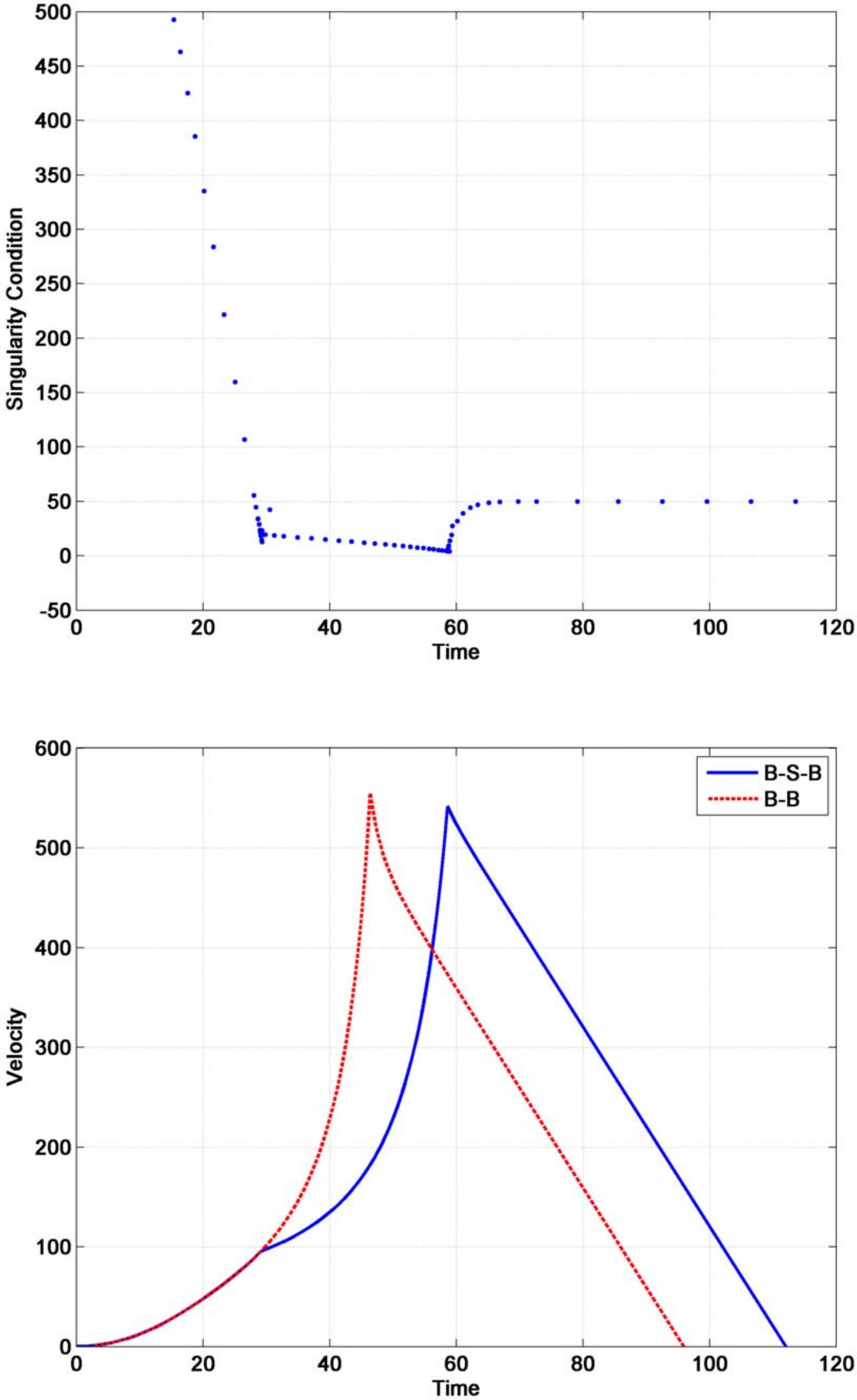

Figure 10.1: Goddard Problem

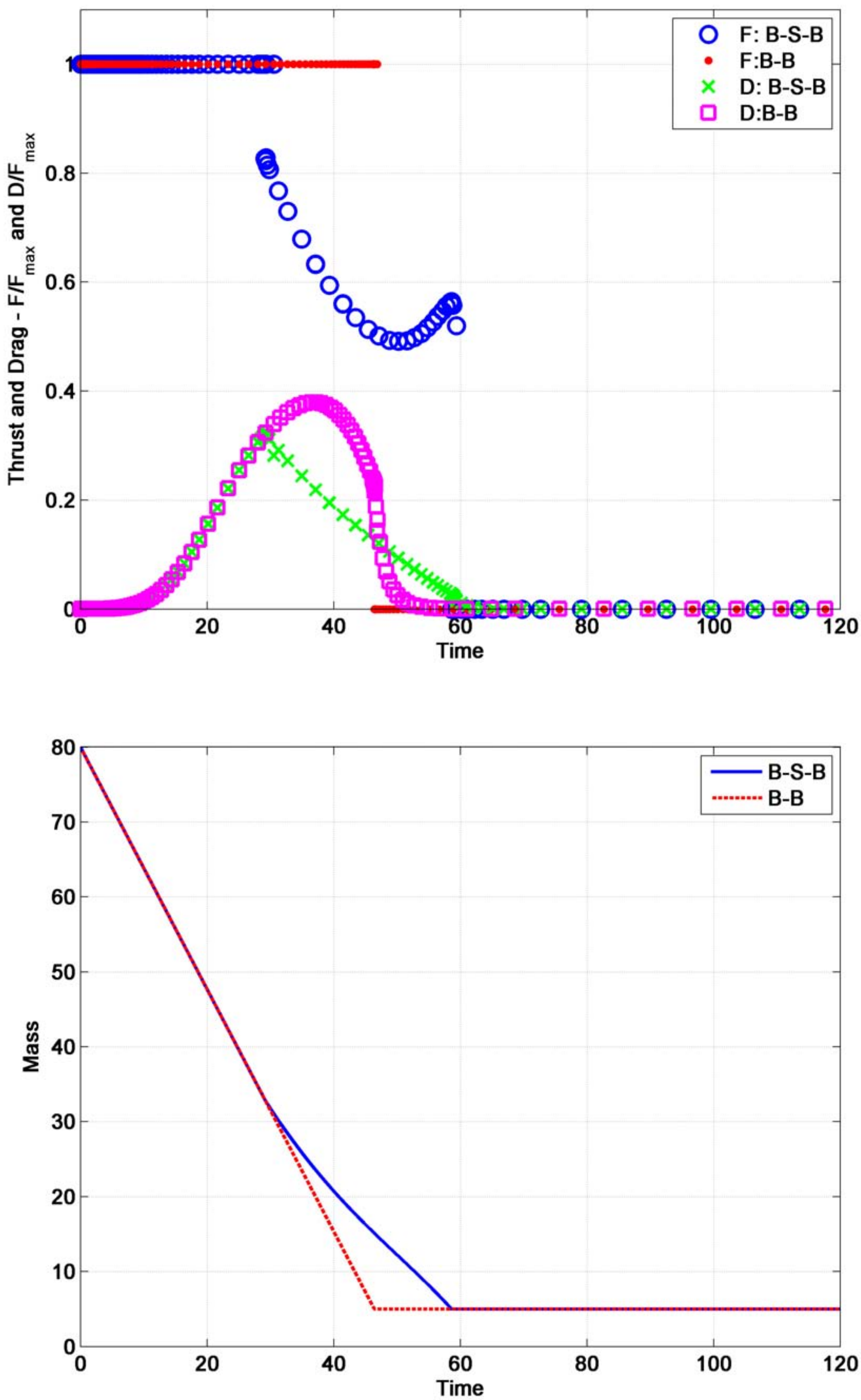

Figure 10.2: Goddard Problem

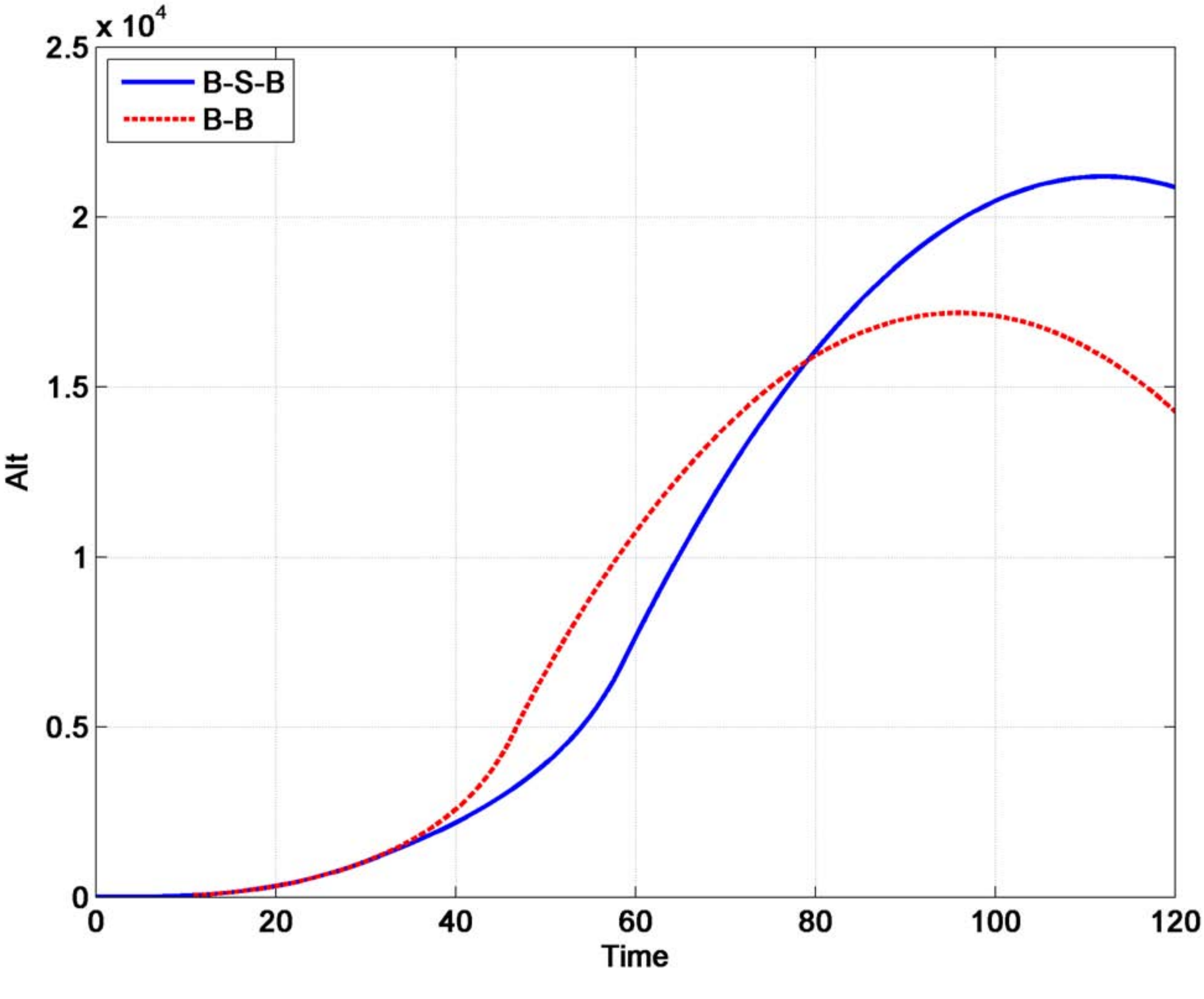

Figure 10.3: Goddard Problem