

# Recitation 11: Time delays

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## 1 Introduction and motivation

1. Delays are incurred when the controller is implemented on a computer, which needs some time to compute the appropriate control input, given a certain error. More in general, the evaluation of sensory information aimed at deciding the best course of action, will require a finite computation time.
2. In some systems, delays may also be part of the physical plant. For example, on an airplane, the effect of lift variation at the main wing are felt on the tail when the vortices shed by the wing reach the tail plane. A rather extreme example is remote tele-operation: communication with a deep-space spacecraft or planetary rover may require several minutes.

## 2 The frequency response of a time delay

A time delay is an operator that transforms an input signal  $t \mapsto u(t)$  into a delayed output signal  $t \mapsto y(t)$ , with  $y(t) = u(t - T)$ , where  $T \geq 0$  is the amount of the delay. Clearly, this is a linear operator: you can check easily that the delayed version of a linear combination of signals is equal to the linear combination of the delayed signals. Let us indicate this operator with  $\text{Delay}_T$ .

In order to compute the transfer function of this linear operator, let us compute the Laplace transform of the output as a function of the Laplace transform of the input. We get

$$Y(s) = \mathcal{L}[y(t)] = \int_0^{\infty} y(t)e^{-st} dt = \int_0^{\infty} u(t - T)e^{-st} dt = \int_0^{\infty} u(\tau)e^{-s\tau - sT} d\tau = e^{-sT}U(s),$$

from which we conclude that

$$\text{Delay}_T(s) = e^{-sT}.$$

What is the frequency response of the time delay? We need to study the function  $\omega \mapsto \text{Delay}_T(j\omega) = e^{-j\omega T}$ . Recall that the polar representation of a complex number  $z \in \mathbb{C}$  is  $z = |z|e^{-j\angle(z)}$ , from which we deduce immediately that

$$|e^{j\omega T}| = 1, \quad \angle(e^{-j\omega T}) = -\omega T.$$

This is not unexpected, since the time-delayed version of a sinusoid of unit amplitude and zero phase  $u(t) = \sin(\omega t)$  is another sinusoid,  $y = \sin(\omega(t - T))$ , with unit magnitude and a phase delayed by  $\omega T$ . (The peak of the input is reached at times  $t_p = \pi/(2\omega) + 2\pi n$ , where the phase of the input is equal to  $\pi/2$ ; at those times, the phase of the output is  $\pi/2 - \omega T$ .) See Figure 1.

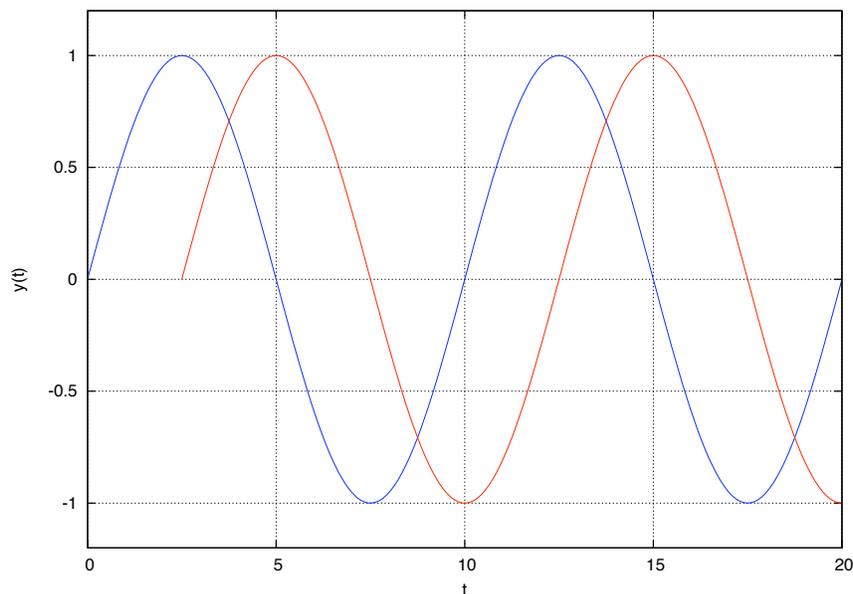


Figure 1: Time-delayed sinusoid. The blue line represents the signal  $y(t) = \sin(2\pi/10t)$  (for  $t \geq 0$ ); the red line is the same signal delayed by 2.5 time units, i.e.,  $\text{Delay}_{2.5}[y]$ . Since  $\omega = 2\pi/10$  rad/s, and  $T = 2.5s$ ,  $\omega T = \pi/2$ : the phase of the time-delayed signal is lagging that of the original signal by exactly  $\pi/2$  radians, or  $90^\circ$  degrees.

On the complex plane, the polar plot of  $e^{-j\omega T}$  corresponds to a circle of unit radius. The Bode magnitude plot is a straight horizontal line at 1, or 0 dB. The Bode phase plot would be a straight line with a negative slope  $-T$ , if we were to plot the phase of the frequency response on a linear scale for the frequency  $\omega$ . However, since in the Bode plot the frequency axis is logarithmic, the Bode phase plot of a time delay is in fact an exponential  $\angle(\text{Delay}_T(s)) = -T 10^{\log_{10} \omega}$  [rad]. To further complicate things, the phase is usually wrapped to  $(-2\pi, 0]$  in Bode plots, so the Bode phase plot of a time delay is usually portrayed as in Figure 2.

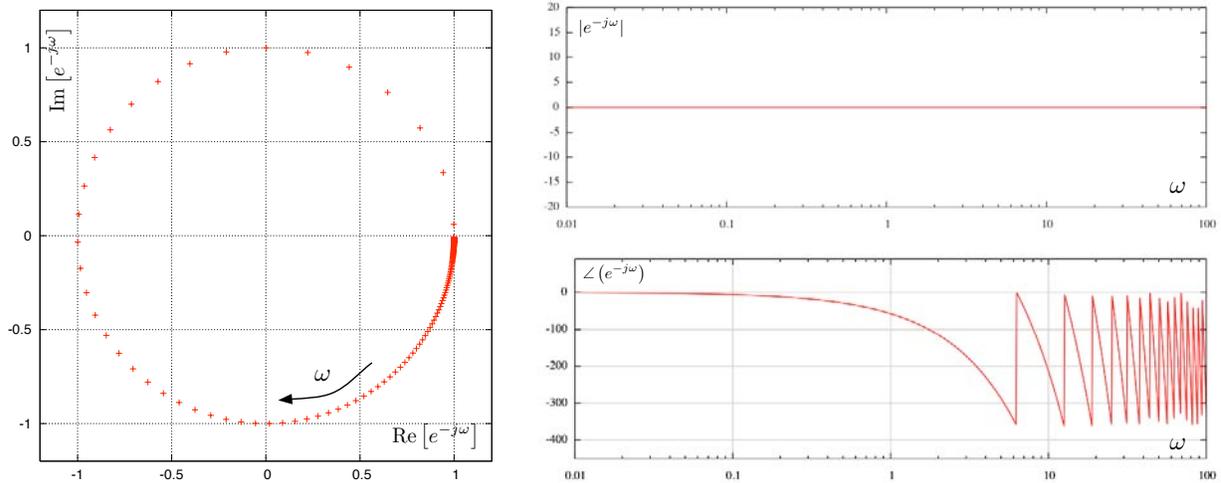


Figure 2: Polar and Bode plots of a unit time delay,  $e^{j\omega}$ .

### 3 Effects of time delays on closed-loop stability

Next, we are going to consider the effects that time delays have on closed-loop stability, and discuss methods to take such effects into account when designing feedback control systems.

Let us consider a system with loop transfer function  $L(s) = C(s)P(s)$ , and include a time delay of  $T$  seconds (this can be either in the controller, or in the plant). We would get a new loop transfer function

$$L'(s) = e^{-sT} L(s).$$

The frequency response of the system with the time delay is obtained from the “ideal” frequency response with no time delay, by shifting the phase back by  $\omega T$ . In other words,

$$|L'(j\omega)| = |L(j\omega)|, \quad \angle L'(j\omega) = \angle L(j\omega) - \omega T, \quad \forall \omega > 0.$$

The effects of such a phase lag are better appreciated with an example. Let us consider a simple plant,

$$P(s) = \frac{1}{s+1},$$

with a proportional controller  $C(s) = k$ . Hence, the ideal loop transfer function is

$$L(s) = \frac{k}{s+1}.$$

A quick check with the root locus or the Nyquist plot will show that the feedback system would be stable for all  $k > 0$  (in fact, for all  $k > -1$ ).

What is the effect of time delay? Consider now the loop transfer function

$$L'(s) = e^{-sT} \frac{k}{s+1}.$$

The polar plot of  $L'$  is obtained from the polar plot of  $L$  by “smearing” it: in practice all points  $L(j\omega)$  must be rotated clockwise about the origin by an angle  $\omega T$ . In other words, the magnitude of such rotation is larger at higher frequencies. In practice, the polar plot of a system with a time delay always looks like a logarithmic spiral close to the origin. See Figure 3.

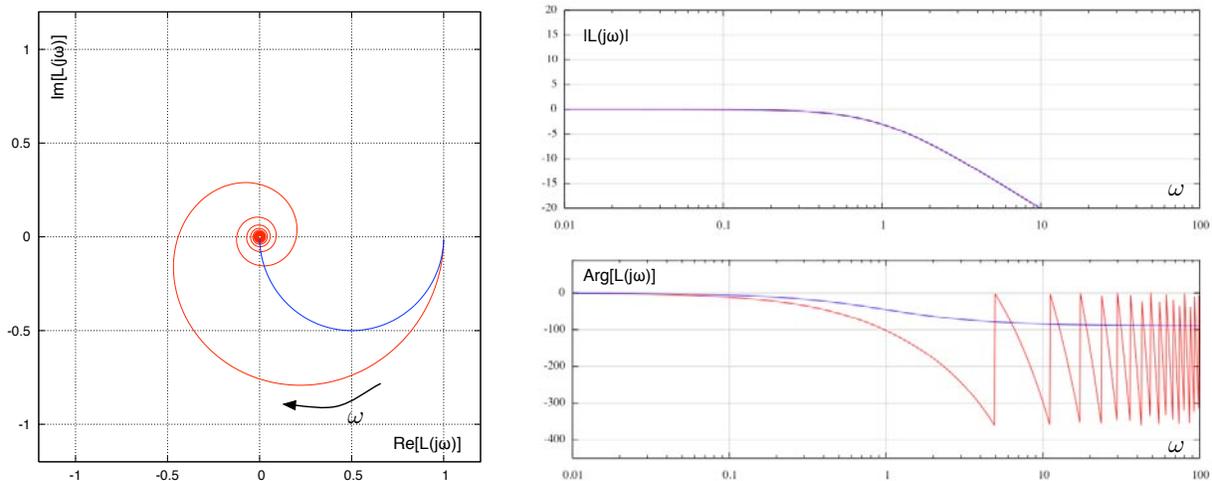


Figure 3: Polar and Bode plots of a simple lag, with and without a unit time delay.

It is easy to recognize that while the polar plot of  $L$  never crosses the negative real axis, the polar plot of  $L'$  crosses it an infinite number of times. In particular, the “first” crossing occurs when

$$\angle \left( \frac{k}{j\omega + 1} \right) - \omega T = -180^\circ,$$

i.e., when

$$\tan^{-1} \omega + \omega T = 180^\circ.$$

The location of this point determines the gain margin, which will be finite for all  $T > 0$  (it was infinite in the ideal case  $T = 0$ ).

For example, in the unit delay case shown in the figures, in which  $T = 1$  seconds, you can verify that the intersection occurs when  $\omega \approx 2.03$  at the point  $(-0.442k, 0)$  on the complex plane. Clearly, if the gain  $k$  were to be greater or equal to  $1/0.0442 = 2.26$ , the Nyquist plot will encircle the  $(-1, 0)$  point, and the closed-loop system would be unstable.

**Bode plot** You can carry out a similar analysis on the Bode plot. First, notice that the magnitude plot of  $L'$  is exactly the same as the magnitude plot of  $L$ . The phase plot is obtained by adding the phase plot of  $L$  to the phase plot of the time delay. This lets you determine very easily the effect of a time delay on the phase margin. In fact, the following relationship holds:

$$\phi_{m,T} = \phi_0 - \omega_c T, \tag{1}$$

where  $\phi_{m,T}$  and  $\phi_{m,0}$  are the phase margins with and without the time delay, respectively, and  $\omega_c$  is the crossover frequency (which does not depend on the time delay). The above equation summarizes the main effect of a time delay, that is a *reduction of the phase margin*. Moreover, *the phase margin reduction increases as the crossover frequency increases*.

If we consider again the example we considered for the Nyquist plot, with the same time delay, you will notice that, while the phase of  $L$  never drops below  $-180$  degrees, the phase of  $L'$  does, for  $\omega > 2.03$  rad/s. This means that in this case the gain  $k$  cannot be increased beyond the point at which crossover occurs at  $\omega_c = 2.03$  rad/s, and we recover the condition  $k < 2.26$  for closed-loop stability.

**A design procedure:** Equation (1) suggests a procedure to design a feedback control system in the presence of time delay, as follows:

1. Design a feedback control system *ignoring* the time delay.
2. Check the effective phase margin, using (1). If the effective phase margin is too small (or negative, indicating closed-loop instability), redesign the controller according to one or both of these criteria:
  - Increase the phase at crossover, e.g., using a phase lead controller.
  - Decrease the crossover frequency, e.g., reducing the gain, or possibly a phase lag controller (to maintain command-following performance).
3. Iterate until a satisfactory controller is found.

## 4 Time delays and the root locus method

So far, we haven't talked about the root locus method in conjunction with time delays. The reason is simple: in order to apply the root locus method, the loop transfer function must be rational, i.e., the ratio of two polynomials in  $s$ . This is not the case when a time delay is present since the term  $e^{-sT}$  is not a rational function.

In order to use the root locus method in this case, it is required to approximate the time delay with a rational transfer function. A first choice would be a Taylor series expansion, which will take the form

$$e^{-sT} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{6}(sT)^3 \dots$$

Let's truncate this series and maintain only the terms up to the second order in  $(sT)$ , i.e., let us write

$$e^{-sT} \approx 1 - sT + \frac{1}{2}(sT)^2.$$

Notice that this is a non-proper transfer function with two non-minimum phase zeros. This would be a good approximation for  $|sT| \ll 1$ . However, the magnitude of the frequency

response diverges for  $\omega \rightarrow \infty$ —whereas we know that the magnitude of  $e^{-j\omega T}$  is always equal to one.

A better approximation can be obtained by using what is called a *Padè approximant*. What we will do is approximate the exponential representing the time delay with a ratio of two polynomials. For simplicity, let us limit ourselves to the ratio of first-order polynomials in  $s$ , i.e., let us write

$$e^{-sT} \approx k \frac{s+p}{s+q}.$$

In order to figure out the right values of the coefficients  $k$ ,  $p$ , and  $q$ , let us ensure that the Padè approximation matches the relevant terms in the Taylor series expansion. In other words, let us impose that

$$k \frac{s+p}{s+q} = 1 - sT + \frac{1}{2}(sT)^2 - \frac{1}{6}(sT)^3 \dots,$$

at least up to the terms that we can match using our three free parameters. We get

$$ks + kp = s + q - s^2T - qsT + \frac{1}{2}s^3T^2 + \frac{1}{2}q(sT)^2 + \dots$$

We can't do anything about the cubic term and higher, but we can ensure that all the terms of order up to and including 2 match, by imposing the following choices:

$$\begin{aligned} \text{Order 0} & : kp = q, \\ \text{Order 1} & : k = 1 - qT, \\ \text{Order 2} & : 0 = -T + \frac{1}{2}qT^2. \end{aligned}$$

Solving for  $k$ ,  $p$ , and  $q$ , we get

$$\begin{aligned} q & = \frac{2}{T}, \\ k & = -1, \\ p & = -\frac{2}{T}, \end{aligned}$$

and finally

$$e^{-sT} \approx \frac{2/T - s}{2/T + s}.$$

Such an approximation has the advantage that the magnitude of its frequency response is always equal to one.

Using the Padè approximation, we can represent a time delay on the root locus as a pole and zero, respectively at  $\pm 2/T$ . Notice the presence of a non-minimum-phase zero.

This approximation method is useful since it allows us to use the root locus method for control design, and gain the insight provided by it. In particular, notice the fact that we can immediately conclude that we cannot increase the gain arbitrarily, since eventually the closed-loop pole will converge to the non-minimum phase zero for large gains.

Considering again the example introduced in the previous sections, the root locus sketch will look like Figure 4. The root locus analysis will tell us that the closed-loop system will remain stable as long as  $k < 3$ . This is not correct, but only gives us a rough estimate of the gain margin. A better approximation may be achieved using a higher-order Padè approximation.

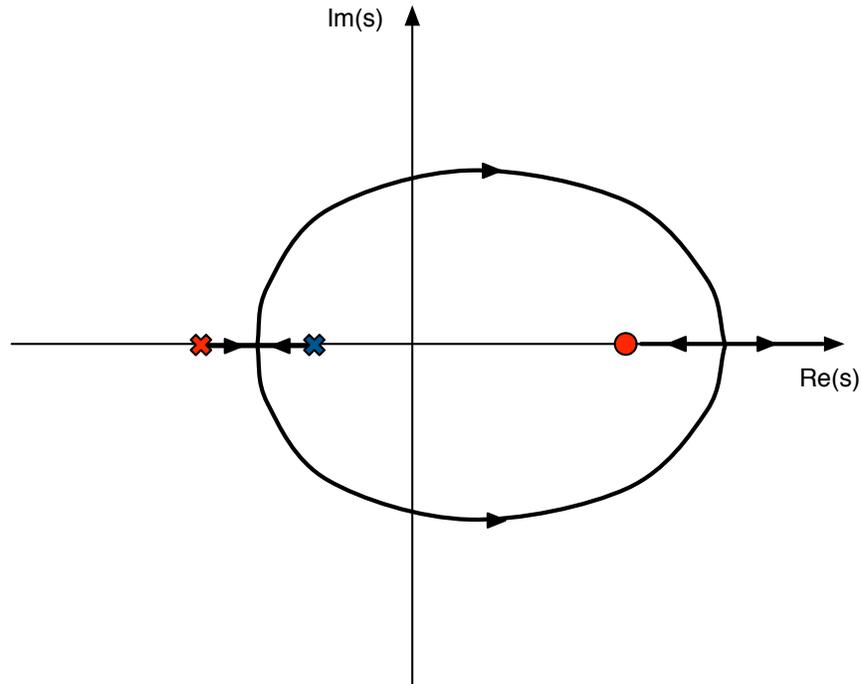


Figure 4: Root locus sketch for a simple lag, with a unit time delay modeled using a first-order Padè approximant.

Summarizing, the Padè approximation allows the use of the root locus method to study the closed-loop stability of a system in the presence of a time delay. However, the results may not be accurate, and it is recommended that the root-locus method be used only as a back-of-the-envelope tool providing some additional insight in the control design process.

**Note:** The *only* tool that would always provide you with correct answers in all cases when a time delay is present (including, e.g., unstable open-loop poles and non-minimum phase zeros) is the Nyquist plot.

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