

## 16.30/31, Fall 2010 — Recitation # 2

September 22, 2010

In this recitation, we will consider two problems from Chapter 8 of the Van de Vegte book.

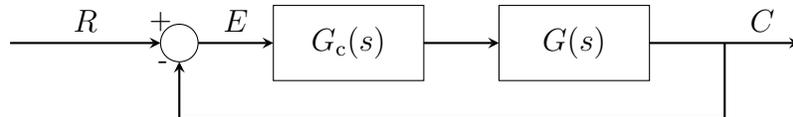


Figure 1: The standard block diagram for a unit-feedback loop.

### Problem 1 (VDV 8.14)

If Figure 1 models a radar tracking system with  $G = 1/[(0.1s + 1)s]$ , design series compensation to meet the following specifications:

1. The steady-state error following ramp inputs must not exceed 2%.
2. The error in response to sinusoidal inputs up to 5 rad/sec should not exceed about 5%.
3. The crossover frequency should be about 50 rad/sec to meet bandwidth requirements while limiting the response to high-frequency noise.
4. The ratio of the break frequencies of  $G_c$  should not exceed 5 to limit noise effects.
5. The phase margin should be about  $50^\circ$ .

**Solution** — The very first thing we should do is check closed-loop stability. A quick root locus sketch will reveal that the closed-loop system is stable for all positive gains, i.e., it is stable for  $G_c(s) = K$ , for  $K > 0$ . For simplicity, we can choose  $K = 1$  to get a baseline stabilizing controller.

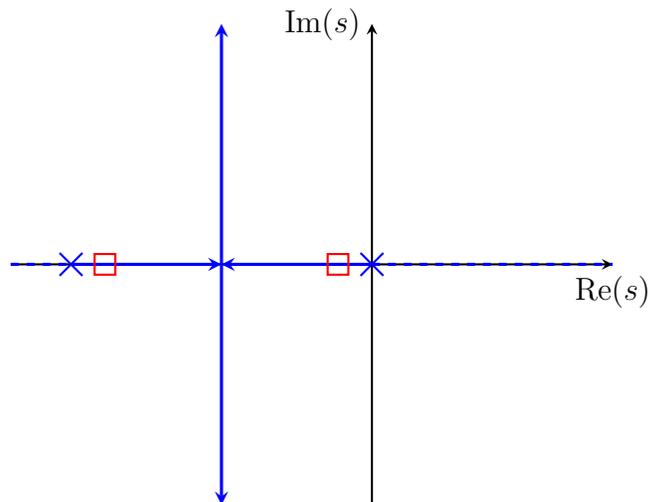


Figure 2: Root locus sketch

Just for practice, let us figure out where the closed-loop poles would be for  $K = 1$ , using root-locus techniques. First of all, write the transfer function in root-locus form, i.e.,  $G(s) = 10 \frac{1}{s(s+10)}$ . Applying the magnitude condition, for  $G_c = K = 1$ , we know that the closed-loop poles will be at locations that satisfy  $|s| \cdot |s + 10| = 10$ . Let us check whether the closed-loop poles would be on the part of the root locus lying on the real axis. In such a case,  $s$  would be a negative real number, and the magnitude condition would read  $-s(s+10) = 10K$ , i.e.,  $s^2 + 10s + 10 = 0$ , which has the two roots  $s = -5 \pm \sqrt{25 - 10} = \{-8.8730, -1.1270\}$ .

Let us sketch the Bode plot of such a system (recall we chose  $G_c = 1$  for the time being, for convenience). The straight-line approximation is shown in Figure 3. Since we have an integrator, the magnitude Bode plot starts with a negative “unit” slope (i.e., -20dB/decade). It will cross the line  $\omega = 1$  at  $K = 1$ . The effect of the pole at  $s = 10$  is to decrease the slope to -40 dB/decade, starting at the break frequency  $\omega = 10$ . The phase will start at -90 degrees for low frequencies (because of the integrator), be equal to -135 degrees at the pole break frequency  $\omega = 10$ , and will ultimately approach -180 degrees at high frequencies (because of the additional pole). Since the phase never drops below -180 degrees, the phase margin will always be positive—but will be small if crossover occurs at high frequencies.

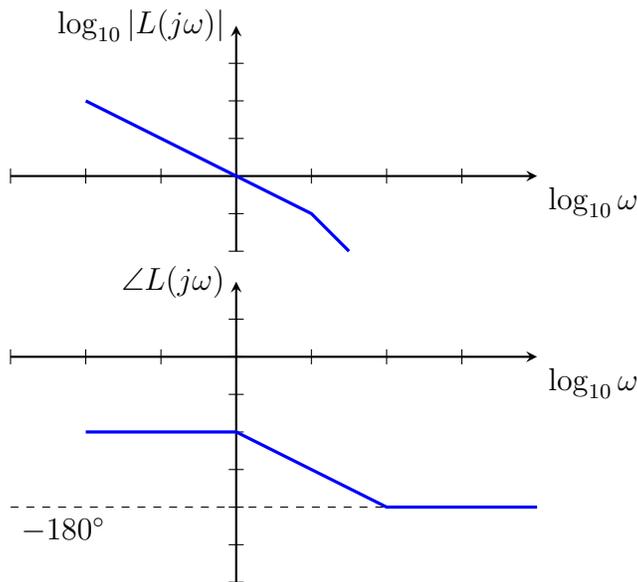


Figure 3: Bode plot for the baseline stabilizing controller

Having established closed-loop stability, let us take a look at the design requirements, one by one.

**1. The steady-state error following ramp inputs must not exceed 2%.**

This requirements says that the system must be of Type 1, i.e.,  $L(s) = G_c(s)G(s)$  must have one pole at the origin (i.e., an integrator), and that, furthermore, the Bode-plot gain of  $L(s)$  must be such that  $e_{ss} = 1/K_B \leq 0.02$  (very often, this is called the “velocity gain”).

Since  $G(s)$  already contains a pole at the origin, there is no need to change the structure of the compensator  $G_c(s) = K$ . However, choosing  $G_c(s) = 1$  would not satisfy the max error requirement—we need the compensator gain to be greater than  $1/0.02$ , i.e.,  $K \geq 50$ . The Bode plot for  $K = 50$  is shown in Figure 4—it is obtained from that in Figure 3 by translating the magnitude plot up by  $\log_{10} 50$ . The phase plot remains unchanged.

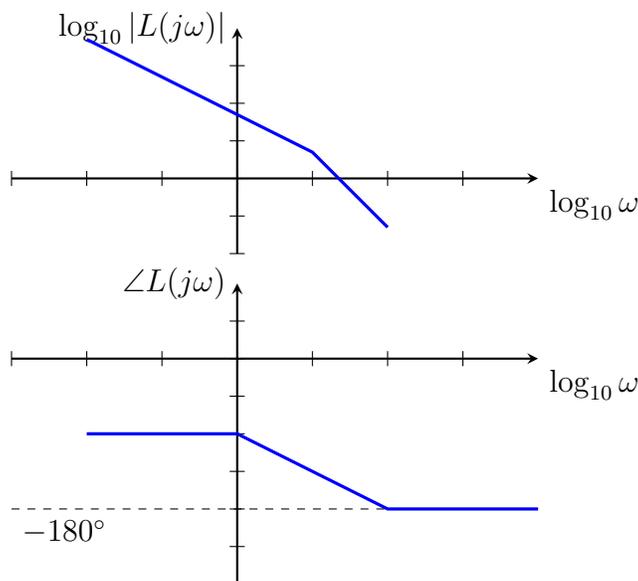


Figure 4: Bode plot for a proportional controller satisfying specification (1).

**2. The error in response to sinusoidal inputs up to 5 rad/sec should not exceed about 5%.**

The error in response to sinusoidal inputs can be evaluated using the frequency response. The transfer function from the reference input to the error (often called the *sensitivity* of the system) is given by

$$S(s) = \frac{1}{1 + L(s)}.$$

In particular, the magnitude of the error in response to a unit-amplitude sinusoid of frequency  $\omega$  is given by  $|S(j\omega)| = 1/|1 + L(j\omega)|$ . Hence, the specification can be written as

$$|1 + L(j\omega)| \geq 1/0.05 = 20, \quad \text{for all } \omega \leq 5\text{rad/s}.$$

In order to satisfy this equation,  $|L(j\omega)|$  should be “large” with respect to 1 (roughly, it should be around 19-21), hence we approximate this condition as

$$|L(j\omega)| \geq 20, \quad \text{for all } \omega \leq 5\text{rad/s}.$$

In other words, the magnitude Bode plot of  $L(s)$  should stay above the “obstacle” shown in Figure 5.

With the previous design choice, the magnitude Bode plot would indeed hit the obstacle. We need to increase the gain in such a way that  $K/5 \geq 20$ , i.e.,  $K \geq 100$ . (Notice that we are using the straight-line approximation... so the real Bode plot would still violate the specification. Still, such an approximation is generally acceptable.)

**The crossover frequency should be about 50 rad/sec to meet bandwidth requirements while limiting the response to high-frequency noise.**

With the current design ( $G_c(s) = K = 100$ , the crossover frequency occurs at approximately  $\omega_c = 31.623$ . You can get this by careful drawing, or through simple geometrical arguments and calculations: the straight line magnitude plot goes through  $|L(j\omega)| = 10$  at  $\omega = 10$ , and

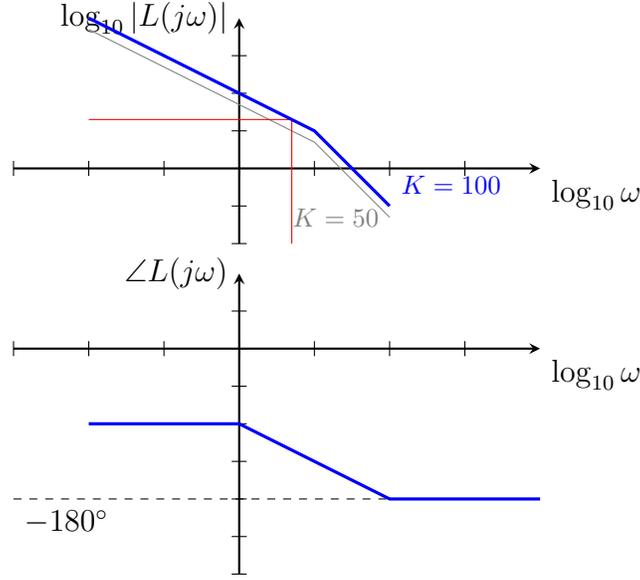


Figure 5: Bode plot for a proportional controller satisfying specification (2).

at that frequency the slope changes to -2, i.e., -40db/decade. In order for the magnitude to decrease further by a factor of 10 (i.e., to reach crossover), the frequency must increase by a factor of  $\sqrt{10} = 3.1623$ .

Using similar arguments, one can recognize that the magnitude of the frequency response, given the current design, is equal to  $10/25=0.4$  at the desired crossover frequency. (The frequency increases by a factor of 5, hence the magnitude decreases by a factor of  $5^2 = 25$  from the “knee” of the Bode plot.)

Hence, in order to make sure that the crossover frequency is exactly at 50 rad/s, we need to multiply  $K$  by a  $1/0.4 = 2.5$ . In other words, choosing  $K = 250$  would satisfy requirements 1-3.

#### 4. The ratio of the break frequencies of $G_c$ should not exceed 5 to limit noise effects.

Hmm. We don’t really need a dynamic compensator up to this point. Let’s set this requirement aside for the time being.

#### 5. The phase margin should be about $50^\circ$ .

While the choice of  $G_c = K = 250$  satisfies requirements 1-3, the phase margin is very small. Using the straight-line approximation, the phase would decrease linearly from  $-90^\circ$  to  $-180^\circ$  as  $\log_{10} \omega$  goes from 0 ( $\omega = 1$  rad/s) to 2 ( $\omega = 100$  rad/s). Hence the phase at the crossover frequency  $\omega_c = 50$  rad/s would be about  $-90 - 90 \log_{10}(50)/2 = -166.45^\circ$ , yielding a phase margin of about  $13.5^\circ$ .

Taking a hint from the requirement (4), we can use a lead compensator to increase the phase at crossover frequency, i.e., consider a compensator of the form

$$G_c(s) = K \frac{\frac{s}{z} + 1}{\frac{s}{p} + 1}.$$

The maximum phase lead we can get is limited by the constraint on the ratio  $\alpha = p/z$  of the break frequencies of the compensator imposed by requirement (4). Using the straight-line

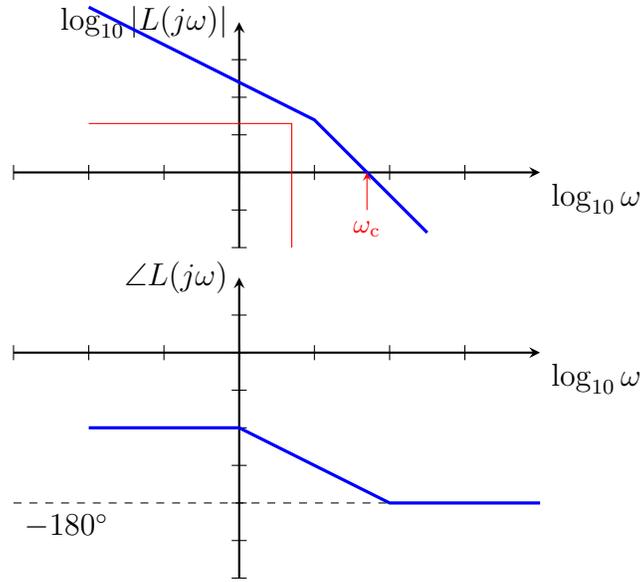


Figure 6: Bode plot for a proportional controller satisfying specification (3).

approximation, we know that the phase lead will be 0 degrees one decade below the zero break frequency, and will increase linearly with  $\log_{10} \omega$  at a rate of 45 degrees/decade until one decade below the pole break frequency, at which point it will remain flat and eventually decrease. In other words, an estimate of the maximum phase lead is  $\phi_{\max} = 45^\circ \log_{10}(5) = 31.45^\circ$ . More accurate calculations, e.g., using the formulas given in class and in the lecture notes, yield a max phase lead of about  $40^\circ$ .<sup>1</sup>

In other words, a lead compensator, if used correctly, could increase the phase margin to the 45-50 degrees range, as desired. How do we get the maximum phase lead contribution to the phase margin? We can achieve this by placing the midpoint (on a log scale, i.e., the geometric mean) of the compensator zero/pole break frequencies exactly at the desired crossover frequency. Doing the math, we get

$$\sqrt{|z| \cdot |p|} = |z| \sqrt{\alpha} = 50 \text{ rad/s},$$

i.e.,  $z = 50/\sqrt{5} = 22.36 \text{ rad/s}$ , and  $p = 5z = 50\sqrt{5} = 111.8 \text{ rad/s}$ .

The resulting straight-line Bode plot is shown in Figure 7. The phase at 50 rad/s is as desired... however, the crossover frequency has moved to the right as an effect of the increased gain at high frequencies due to the lead compensator.

In order to recover the correct crossover frequency, and the desired phase margin, we can lower the gain (i.e., shift the magnitude Bode plot downwards). Fortunately, we have some room to reduce the gain and still meet requirements (1) and (2), since we artificially increased the gain to 250 in step (3).

According to the straight-line approximation, if  $G_c(s) = 250(s/22.36 + 1)/(s/111.8 + 1)$ , the magnitude of  $L(j\omega)$  at  $\omega = 50 \text{ rad/s}$  can be obtained as follows:

- $L(j1) = 250$ .
- $L(j10) = L(j1)/10 = 25$  (-1 slope).

<sup>1</sup>I apologize for giving an incorrect number in class, stating that this max phase lead would be about 60 degrees.

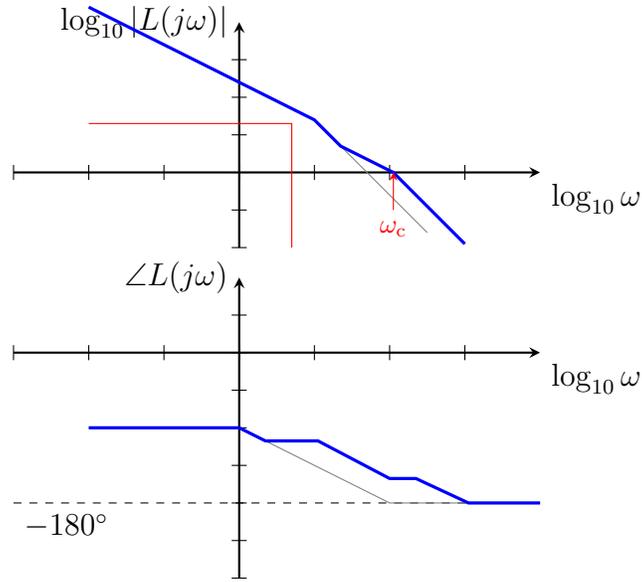


Figure 7: Bode plot for a candidate lead compensator satisfying specifications (4) and (5). The crossover frequency is higher than desired, and the phase margin is smaller.

- $L(j22.36) = L(j10)/(22.36/10)^2 = 5$  (-2 slope).
- $L(j50) = L(j22.36)/(50/22.36) = 2.236$  (-1 slope).

In other words, we need to reduce the gain by a factor of 2.236, i.e., set  $K = 250/2.236 = 111.8$ , and finally set

$$G_c(s) = 111.8 \frac{\frac{s}{22.36} + 1}{\frac{s}{111.8} + 1},$$

or, in the equivalent root-locus form,

$$G_c(s) = 559 \frac{s + 22.36}{s + 111.8}.$$

Note that this compensator satisfies the error specifications (1) and (2), since  $K$  is still greater than the minimum values we obtained from those requirements. If this had not been the case, we could have added a lag compensator at low frequencies.

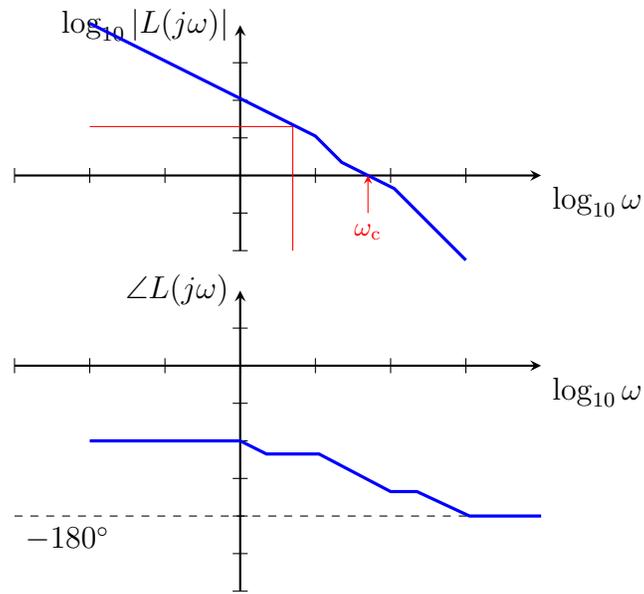


Figure 8: Bode plot for the final lead compensator satisfying specifications (1)-(5).

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