

16.30/31, Fall 2010 — Recitation # 1

September 14, 2010

In this recitation we considered the following problem. Given a plant with open-loop transfer function

$$G_p(s) = -\frac{1.1569s + 0.3435}{s^2 + 0.7410s + 0.9272},$$

design a feedback control system such that the closed-loop dominant poles have undamped natural frequency $\omega_n = 3\text{rad/s}$ and damping ratio $\zeta = 0.6$. (This problem was mentioned during lecture, see the Topic 2 notes, as a stability augmentation system for a B747.)

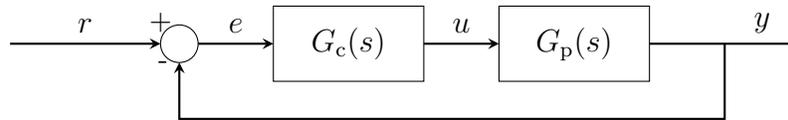


Figure 1: The standard block diagram for a unit-feedback loop.

Since we are working with the root locus method, it is convenient to rewrite the transfer function in the form

$$G_p(s) = K_p \frac{\prod_{z \in \text{Zeros}(G_p)} (s - z)}{\prod_{p \in \text{Poles}(G_p)} (s - p)} = -1.1569 \cdot \frac{s + 0.2969}{s^2 + 0.7410s + 0.9272}.$$

Note that the “root-locus gain” of the plant’s transfer function $K_p = -1.1569$ is *negative*. This is a fact that is easy to overlook, and may generate confusion if not recognized and handled properly. More on this later.

The pole-zero map for the system is shown in Figure 2, where:

- $z_1 = -0.2969$ is the zero (root of the numerator) of G_p .
- $p_{1,2} = -0.3705 \pm 0.8888j$ are the poles (roots of the denominator) of G_p .
- $p_{1,2}^d = -1.8 \pm 2.4j$ are the desired closed-loop poles. (We know that the magnitude of the desired poles is equal to $\omega_n = 3$, and the real part is $-\zeta\omega_n = -1.8$.)

Q1. Can we achieve the control objective using proportional feedback?

A first question one may want to answer is whether a simple proportional feedback (i.e., $G_c(s) = K$, for some K) would do the trick.

You may convince yourself that this is not likely to be the case, e.g., by sketching the root locus. However, it is possible to check rigorously whether a point is on the root locus (i.e., whether the root locus goes through a given point), using the angle condition:

$$\sum_{z \in \text{Zeros}(G)} \angle(s - z) - \sum_{p \in \text{Poles}(G)} \angle(s - p) = l180^\circ, \quad l = 0, \pm 1, \pm 2, \dots$$

(In particular, the point will be on the positive-gain root locus for l odd, and on the negative-gain root locus for l even.)

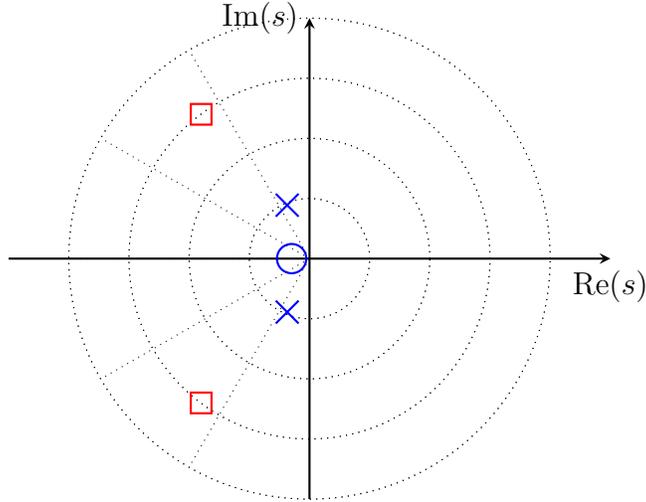


Figure 2: The open loop poles (blue crosses) are in an undesirable locations: the response is too slow ($\omega_n \approx 1\text{rad/s}$) and too lightly damped ($\zeta \approx 0.4$). The objective is to design a control system such that the closed loop system has its dominant poles in the location marked with red squares.

Recall that, e.g., $(s - z)$ can be thought of as a vector going from z to s . The angle $\angle(s - z)$ can be computed as, the arctangent of $\text{Im}(s - z)/\text{Re}(s - z)$. (You may want to use the `atan2` function on your calculator or in Matlab to make sure you get the quadrant right.)

In our case, let us check where the desired poles are on the root locus. Due to symmetry, it is sufficient to check for one of them, i.e., for $p_1^d = -1.8 + 2.4j$. We get

- $p_1^d - z_1 = -1.5031 + 2.4j$, hence $\angle(p_1^d - z_1) = \text{atan2}(2.4, -1.5031) = 122.06^\circ$.
- $p_1^d - p_1 = -1.4295 + 1.5112j$, hence $\angle(p_1^d - p_1) = \text{atan2}(1.5112, -1.4295) = 133.41^\circ$.
- $p_1^d - p_2 = -1.4295 + 3.2888j$, hence $\angle(p_1^d - p_2) = \text{atan2}(3.2888, -1.4295) = 113.50^\circ$.

Hence,

$$\angle(p_1^d - z_1) - \angle(p_1^d - p_1) - \angle(p_1^d - p_2) = -124.84^\circ \neq l180^\circ, \quad l = 0, \pm 1, \pm 2, \dots$$

and the point p_1^d is not on the root locus (for proportional feedback).

Q2. Can we achieve the control objective using a dynamic compensator?

To make a long story short: yes. There are many ways one can achieve the objective. Let us consider one, we will discuss others shortly.

Let us use a dynamic compensator of the form

$$G_c(s) = K_p \frac{s - z_c}{s - p_c}.$$

Let us choose $p_c = z_1$, i.e., let us cancel the open-loop zero with a compensator pole. We still have two degrees of freedom remaining: the location of the zero z_c and the gain K_c .

The angle condition in this case would be (recall that the open-loop zero is canceled with the compensator pole):

$$\angle(p_1^d - z_c) - \angle(p_1^d - p_1) - \angle(p_1^d - p_2) = l180^\circ, \quad l = 0, \pm 1, \pm 2, \dots,$$

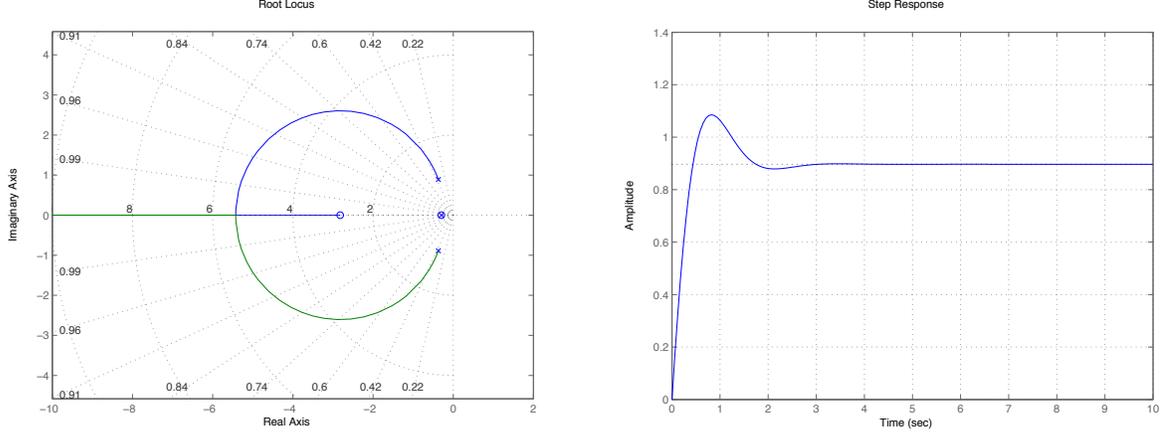


Figure 3: Root locus and closed-loop step response using the lag compensator proposed in the text.

that is (recall that we have already computed the angles of $p_1^d - p_{1,2}$),

$$\angle(p_1^d - z_c) = 246.9^\circ + l180^\circ, \quad l = 0, \pm 1, \pm 2, \dots$$

In other words, if we choose z_c in such a way that $\angle(p_1^d - z_c) = 66.9^\circ$, we satisfy the angle condition, with $l = -1$. Since l is odd, the desired poles will be on the positive-gain root locus for the proposed feedback system.

In order to find z_c , just set

$$\angle(p_1^d - z_c) = \text{atan2}(2.4, -1.8 - z_c) = 66.9^\circ,$$

i.e.,

$$z_c = -1.8 - \frac{2.4}{\tan 66.9^\circ} = -2.82.$$

Notice that given the location of the compensator pole and zero, this is a lag compensator (the pole is closer to the origin). The magnitude ratio between the pole and the zero is within the standard guidelines of a factor between 5 and 10.

The last step in the design of the compensator is the choice of the gain. The magnitude condition can be written as:

$$\frac{\prod_{z \in \text{Zeroes}(G)} |s - z|}{\prod_{z \in \text{Poles}(G)} |s - p|} = \frac{1}{|K|},$$

where $G = G_c G_p$ and $K = K_c K_p$. Doing the calculations (remember we can ignore the canceled pole/zero), we get: $|K| = 2.859$, and ultimately $|K_c| = |K|/|K_p| = 2.471$. Since $K_p < 0$ as already mentioned, we can conclude $K_c = -2.471$, and

$$G_c = -2.471 \frac{s + 2.82}{s + 0.2969}.$$

For verification purposes, we can compute the characteristic equation of the closed-loop system:

$$1 + G_c G_p = \frac{s^2 + 0.7410s + 0.9272 + 2.859(s + 2.82)}{\dots} = \frac{s^2 + 3.6s + 9}{\dots},$$

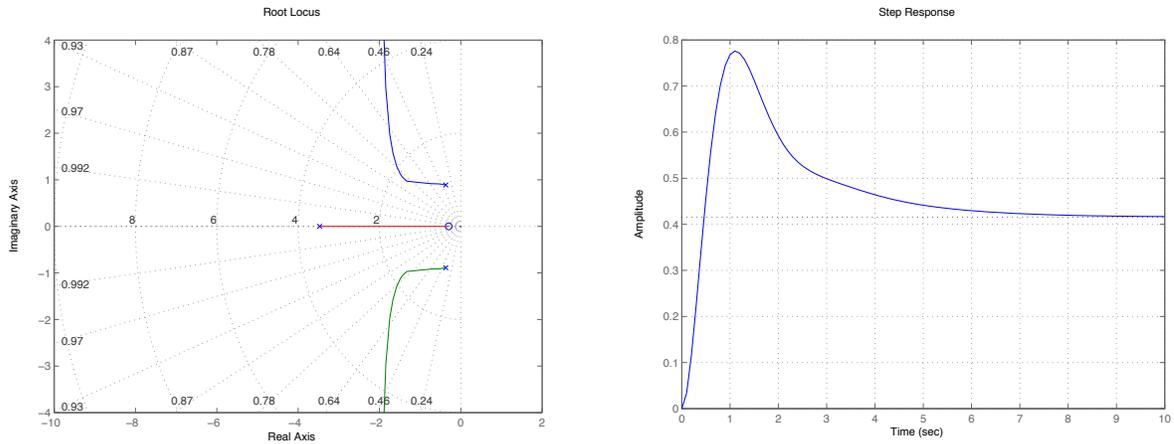


Figure 4: Root locus and closed-loop step response using the simple compensator with one pole discussed during the recitation.

which shows that the closed-loop poles are indeed at the desired locations.

A similar lag-compensation approach would have worked, without the need for pole-zero cancellation, but the algebra would have been a bit more complicated.

The root locus and the (closed-loop) step response using the proposed compensator are shown in Figure 3.

Other approaches

Simple lag One approach proposed in class was consider a dynamic compensator with only one pole p_c . The angle condition can be written as

$$\angle(p_1^d - z_1) - \angle(p_1^d - p_1) - \angle(p_1^d - p_2) - \angle(p_1^d - p_c) = -124.84^\circ - \angle(p_1^d - p_c) = l180^\circ, \quad l = 0, \pm 1, \pm 2, \dots,$$

i.e., we can choose

$$\angle(p_1^d - p_c) = 55.16^\circ,$$

satisfying the angle condition with $l = -1$. Since l is odd, the desired closed loop poles will be on the positive-gain root locus.

Proceeding as in the previous case, we can compute $p_c = -1.8 - 2.4 / \tan 55.16^\circ = -3.47$. The magnitude condition would yield $K_c = -6.658$.

Indeed, two of the closed-loop poles will be at the desired location. However, a new pole has been introduced in the system, and will actually play a dominant role in the response, which is not desirable—see Figure 4. Also note that the steady-state error is quite large.

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