Topic #24

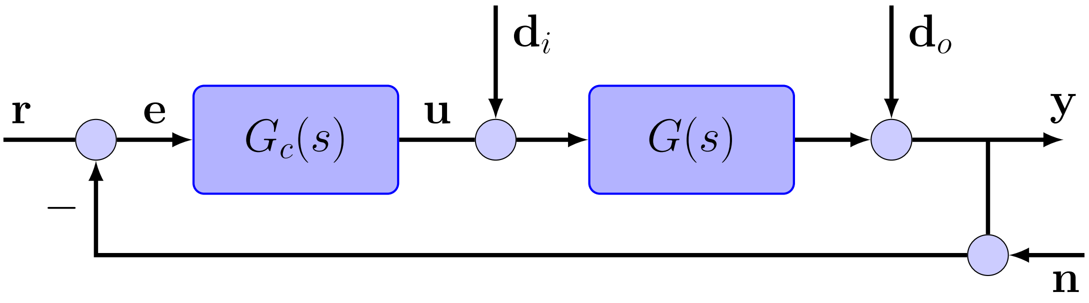
16.30/31 Feedback Control Systems

Closed-loop system analysis

- Bounded Gain Theorem
- Robust Stability

SISO Performance Objectives

- Basic setup:



Where the tracking error can be written as:

$$\begin{aligned} \mathbf{e} &= \mathbf{r} - (\mathbf{y} + \mathbf{n}) \\ &= \mathbf{r} - (\mathbf{d}_o + G(\mathbf{d}_i + G_c\mathbf{e}) + \mathbf{n}) \\ &= S(\mathbf{r} - \mathbf{d}_o - \mathbf{n}) - SG\mathbf{d}_i \end{aligned}$$

$$\mathbf{y} = T(\mathbf{r} - \mathbf{n}) + S\mathbf{d}_o + SG\mathbf{d}_i$$

with

$$L = GG_c, \qquad S = (I + L)^{-1} \qquad T = L(I + L)^{-1}$$

- For good tracking performance of $\mathbf{r}$ (typically low frequency), require $\mathbf{e}$ small

$$\Rightarrow \quad \|S(\mathbf{j}\omega)\| \ \text{ small } \ \forall \ \ 0 \le \omega \le \omega_1$$

- To reduce impact of sensor noise $\mathbf{n}$ (typically high frequency), require

$$\Rightarrow \quad \|T(\mathbf{j}\omega)\| \ \text{ small } \ \forall \ \ \omega \ge \omega_2$$

- Since

$$T(s) + S(s) = I \ \forall \ \ s$$

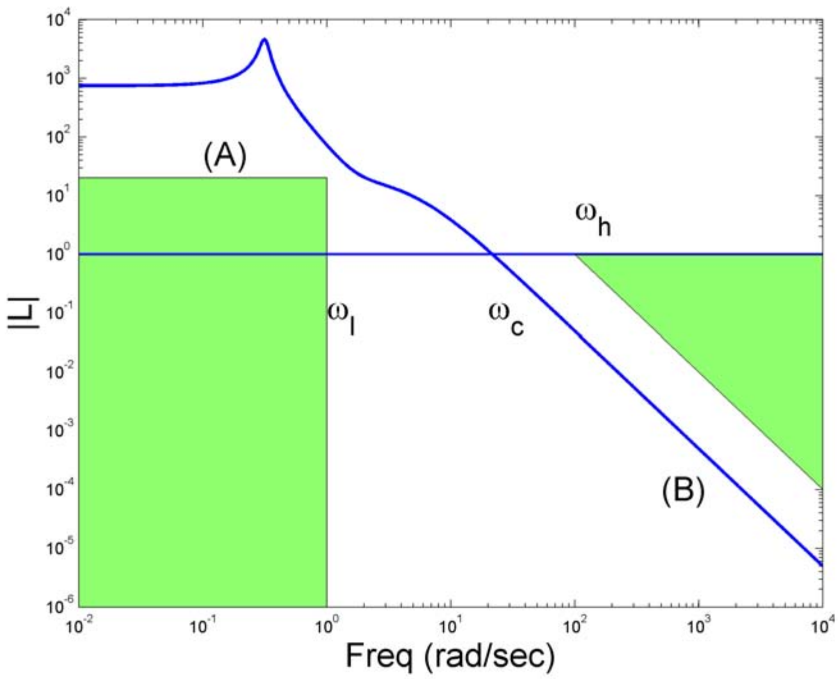
cannot make both $\|S(\mathbf{j}\omega)\|$ and $\|T(\mathbf{j}\omega)\|$ small at the same frequencies $\Rightarrow$ fundamental design constraint

SISO Design Approaches

- There are two basic approaches to design:
  - **Indirect** that works on L itself – classical control does this
  - **Direct** that works on S and T

- For the **Indirect**, note that
  - If $|L(\mathbf{j}\omega)| \gg 1 \Rightarrow S = (1+L)^{-1} \approx L^{-1} \Rightarrow |S| \ll 1$, $|T| \approx 1$
  - If $|L(\mathbf{j}\omega)| \ll 1 \Rightarrow S = (1+L)^{-1} \approx 1$ and $T \approx L \Rightarrow |T| \ll 1$

  So we can convert the performance requirements on S, T into specifications on L



(A) High loop gain $\leadsto$ Good command following & dist rejection
(B) Low loop gain $\leadsto$ Attenuation of sensor noise.

- Of course, we must be careful when $|L(\mathbf{j}\omega)| \approx 1$, require that $\arg L \neq \pm 180^\circ$ to maintain stability

- **Direct approach** works with S and T. Since $e = r - y$, then for perfect tracking, we need $e \approx 0$

$$\Rightarrow \text{ want } S \approx 0 \text{ since } e = Sr + \ldots$$

  - Sufficient to discuss the magnitude of S because the only requirement is that it be small.

- Direct approach is to develop an upper bound for $|S|$ and then *test* if $|S|$ is below this bound.

$$|S(\mathbf{j}\omega)| < \frac{1}{|W_s(\mathbf{j}\omega)|} \qquad \forall \omega?$$

or equivalently, whether $|W_s(\mathbf{j}\omega)S(\mathbf{j}\omega)| < 1, \quad \forall \omega$

- Typically pick simple forms for weighting functions (first or second order), and then cascade them as necessary. Basic one:

$$W_s(s) = \frac{s/M + \omega_B}{s + \omega_B A}$$

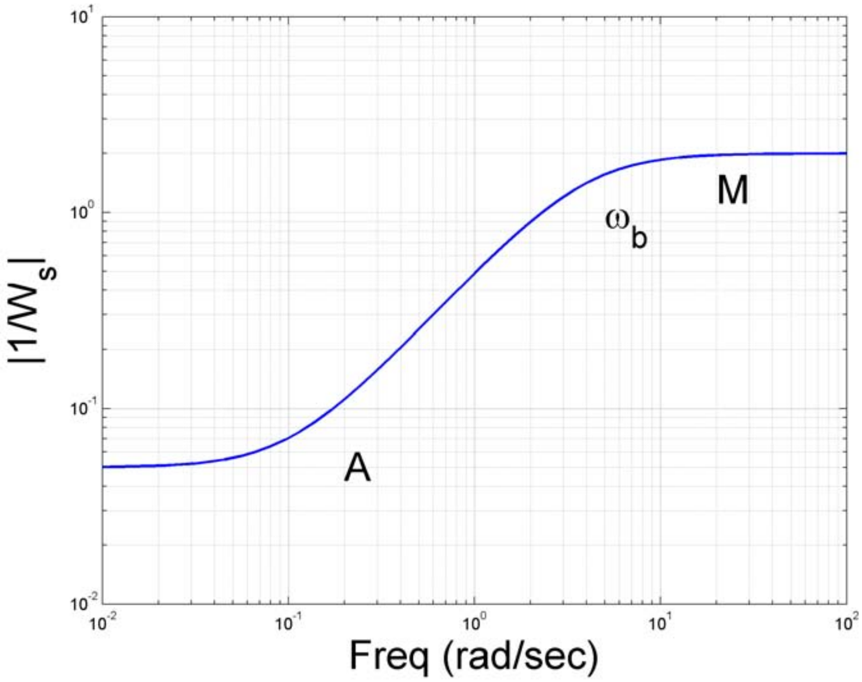


Fig. 1: Example of a standard performance weighting filter. Typically have $A \ll 1$, $M > 1$, and $|1/W_s| \approx 1$ at ω_B

- **Example:** Simple system with $G_c = 1$

$$G(s) = \frac{150}{(10s+1)(0.05s+1)^2}$$

- Require $\omega_B \approx 5$, a slope of 1, low frequency value less than $A = 0.01$ and a high frequency peak less than $M = 5$.

$$W_s = \frac{s/M + \omega_B}{s + \omega_B A}$$

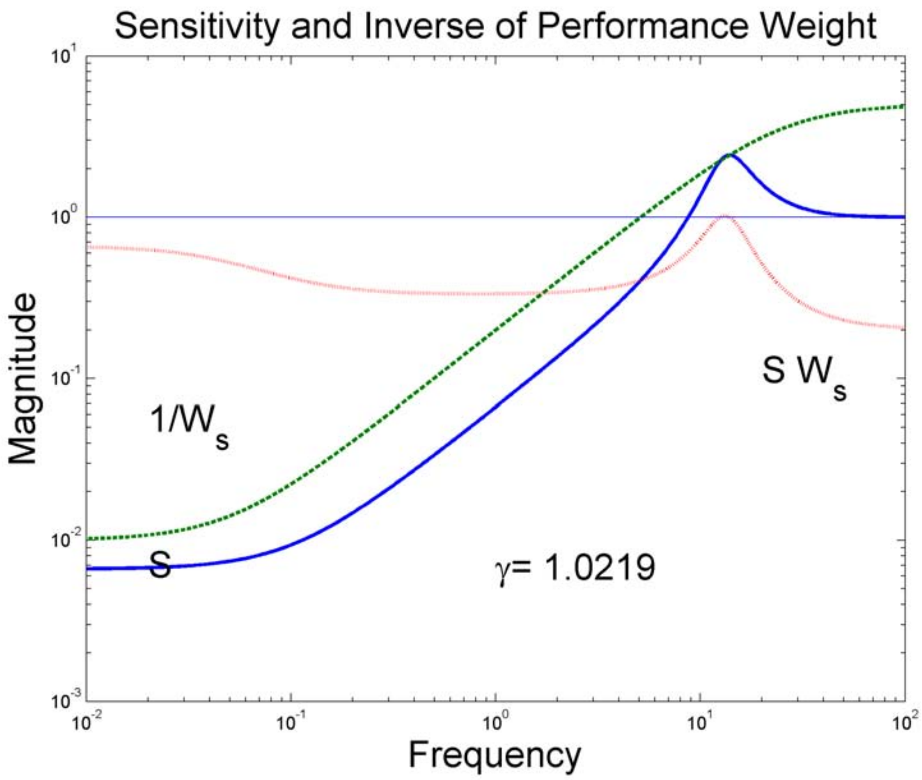


Fig. 2: Want $|SW_p| < 1$, so we just fail the test

- Graph testing is OK, but what we need is a an analytical way of determining whether

$$|W_s(\mathbf{j}\omega)S(\mathbf{j}\omega)| < 1, \quad \forall\omega$$

  - Avoids a graphical/plotting test, which might be OK for analysis (a bit cumbersome), but very hard to use for synthesis

Bounded Gain

- There exist very easy ways of testing (analytically) whether

$$|S(\mathbf{j}\omega)| < \gamma, \quad \forall\omega$$

  - Critically important test for robustness

- **SISO Bounded Gain Theorem:** Gain of generic stable system[1]

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t)\\ y(t) &= C\mathbf{x}(t) + Du(t)\end{aligned}$$

is bounded in the sense that

$$G_{\max} = \sup_{\omega} |G(\mathbf{j}\omega)| = \sup_{\omega} |C(\mathbf{j}\omega I - A)^{-1}B + D| < \gamma$$

if and only if (iff)

1. $|D| < \gamma$

2. The **Hamiltonian matrix**

$$\mathcal{H} = \left[\begin{array}{c|c} A + B(\gamma^2 I - D^T D)^{-1} D^T C & B(\gamma^2 I - D^T D)^{-1} B^T \\ \hline -C^T(I + D(\gamma^2 I - D^T D)^{-1} D^T)C & -A^T - C^T D(\gamma^2 I - D^T D)^{-1} B^T \end{array}\right]$$

has no eigenvalues on the imaginary axis.

[1]MIMO result very similar, but need a different norm on the TFM of $G(s)$

- Note that with $D = 0$, the **Hamiltonian matrix** is

$$\mathcal{H} = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}$$

  - Eigenvalues of this matrix are symmetric about the real and imaginary axis (related to the SRL)

- So $\sup_\omega |G(\mathbf{j}\omega)| < \gamma$ iff $\mathcal{H}$ has no eigenvalues on the $\mathbf{j}\omega$-axis.

  - An equivalent test is if there exists a $X \geq 0$ such that

$$A^TX + XA + C^TC + \frac{1}{\gamma^2}XBB^TX = 0$$

  and $A + \frac{1}{\gamma^2}BB^TX$ is stable.

- This is another **Algebraic Riccati Equation** (ARE)
  - But there are some key differences from the LQR/LQE ARE's

Typical Application

- Direct approach provides an upper bound for $|S|$, so must *test* if $|S|$ is below this bound.

$$|S(\mathbf{j}\omega)| < \frac{1}{|W_s(\mathbf{j}\omega)|} \qquad \forall\omega?$$

or equivalently, whether $|W_s(\mathbf{j}\omega)S(\mathbf{j}\omega)| < 1, \quad \forall\omega$

- Pick simple forms for weighting functions (first or second order), and then cascade them as necessary. Basic one:

$$W_s(s) = \frac{s/M + \omega_B}{s + \omega_B A}$$

- Thus we can test whether $|W_s(\mathbf{j}\omega)S(\mathbf{j}\omega)| < 1, \quad \forall\omega$ by:
  1. Forming a state space model of the combined system $W_s(s)S(s)$
  2. Use the bounded gain theorem with $\gamma = 1$
  3. Typically use a bisection section of γ to find $|W_s(\mathbf{j}\omega)S(\mathbf{j}\omega)|_{\max}$

- For our simple example system

$$G(s) = \frac{150}{(10s+1)(0.05s+1)^2} \qquad G_c = 1$$

  - Require $\omega_B \approx 5$, a slope of 1, low frequency value less than $A = 0.01$ and a high frequency peak less than $M = 5$.
  - In this case $\gamma = 1.02$, so we just fail the test - consistent with graphical test.

Issues

- Note that it is actually not easy to find $G_{\max}$ directly using the state space techniques
  - It is easy to check if $G_{\max} < \gamma$
  - So we just keep changing γ to find the smallest value for which we can show that $G_{\max} < \gamma$ (called $\gamma_{\min}$)

  $\Rightarrow$ Bisection search algorithm.

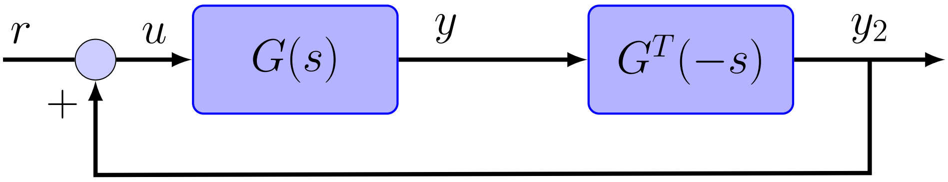
- **Bisection search algorithm** (see web)
  1. Select γ_u, γ_l so that $\gamma_l \leq G_{\max} \leq \gamma_u$
  2. Test $(\gamma_u - \gamma_l)/\gamma_l < \text{TOL}$.
     Yes $\Rightarrow$ Stop ($G_{\max} \approx \frac{1}{2}(\gamma_u + \gamma_l)$)
     No $\Rightarrow$ go to step 3.
  3. With $\gamma = \frac{1}{2}(\gamma_l + \gamma_u)$, test if $G_{\max} < \gamma$ using $\lambda_i(\mathcal{H})$
  4. If $\lambda_i(\mathcal{H}) \in \mathbf{j}\mathbb{R}$, then set $\gamma_l = \gamma$ (test value too low), otherwise set $\gamma_u = \gamma$ and go to step 2.

- This is the basis of $\mathcal{H}_\infty$ control theory.

Appendix: Sketch of Proof

- Sufficiency: consider $\gamma = 1$, and assume $D = 0$ for simplicity

- Now analyze properties of this special SISO closed-loop system.



$$G(s) := \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \quad \text{and} \quad G^{\sim}(s) \equiv G^T(-s) := \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & 0 \end{array}\right]$$

- Note that
$$u/r = \mathcal{S}(s) = [1 - G^{\sim}G]^{-1}$$

- Now find the state space representation of $\mathcal{S}(s)$
$$\begin{aligned} \dot{x}_1 &= Ax_1 + B(r + y_2) = Ax_1 + BB^Tx_2 + Br \\ \dot{x}_2 &= -A^Tx_2 - C^Ty = -A^Tx_2 - C^TCx \\ u &= r + B^Tx_2 \end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$
$$u = \begin{bmatrix} 0 & B^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r$$

$\Rightarrow$ poles of $\mathcal{S}(s)$ are contained in the eigenvalues of the matrix $\mathcal{H}$.

- Now assume that $\mathcal{H}$ has no eigenvalues on the $\mathbf{j}\omega$-axis,

$$\Rightarrow \mathcal{S} = [I - G^{\sim}G]^{-1} \text{ has no poles there}$$

$$\Rightarrow I - G^{\sim}G \text{ has no zeros there}$$

- So $I - G^{\sim}G$ has no zeros on the $\mathbf{j}\omega$-axis, and we also know that $I - G^{\star}G \to I > 0$ as $\omega \to \infty$ (since $D = 0$).
  - Together, these imply that

$$I - G^{\star}G = I - G^{T}(-\mathbf{j}\omega)G(\mathbf{j}\omega) > 0 \;\; \forall \;\; \omega$$

- For a SISO system, condition $(I - G^{\star}G > 0)$ is equivalent to

$$|G(\mathbf{j}\omega)| < 1 \;\; \forall \;\; \omega$$

which is true iff

$$G_{\max} = \max_{\omega} |G(\mathbf{j}\omega)| < 1$$

- Can use state-space tools to test if a generic system has a gain less that 1, and can easily re-do this analysis to include bound γ.

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