16.30/31 Feedback Control Systems

Overview of Nonlinear Control Synthesis

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An overview of nonlinear control design methods

- Extend applicability of linear design methods:
  - Gain scheduling
  - Integrator anti-windup schemes
- Geometric control
  - Feedback linearization
  - Dynamics inversion
  - Differential flatness
- Adaptive control
  - Neural network augmentation
- Lyapunov-based methods/Contraction theory
  - Control Lyapunov Functions
  - Sliding mode control
  - Backstepping
- Computational/logic approaches
  - Hybrid systems
  - Model Predictive Control

Gain scheduling

- Nonlinear system: $\dot{x} = f(x, u)$.
- Choose n equilibrium points, i.e., (x_i^*, u_i^*), such that $f(x_i^*, u_i^*) = 0$, $i = 1, \ldots n$.
- For each of these equilibria, linearize the system and design a "local" control law $u_i(x) = u_i^* - K(x - x_i^*)$ for the linearization.
- A global control law consists of:
  - Choose the right control law, as a function of the state: $i = \sigma(x)$
  - Use that control law: $u(x) = u_{\sigma(x)}(x)$

Control Lyapunov functions

- Nonlinear system: $\dot{x} = f(x) + g(x)u$, with equilibrium at $x = 0$
- A function $V : x \mapsto V(x)$ is a Control Lyapunov Function if
  - It is positive definite
  - $V(0) = 0$.
  - It is always possible to find u such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x)u \leq 0.$$

- If V is a CLF, it is always possible to design a control law ensuring $\dot{V} \leq 0$!

Differential Flatness

- A dynamical system

$$\frac{dx}{dt} = f(x, u),$$
$$z = h(x, u).$$

is said to be differentially flat, with flat output z, if one can compute the state and input trajectories as a function of the flat outputs and a finite number of its derivatives, i.e., if one can find a map Ξ such that

$$(x, u) = \Xi(z, \dot{z}, \ldots, z^{(l)}).$$

- Differential flatness can be shown to be equivalent to (dynamic) feedback linearizability. (But more intuitive.)

A differentially-flat model of aircraft dynamics 1/4

- Given a reference trajectory, p_d, we can compute the reference velocity $\dot{p}_d$, and the reference acceleration $\ddot{p}_d$.
- Based on these, and assuming coordinated flight (and hence $\dot{p}_d \neq 0$) we can get a set of reference wind axes:
  - The x_w axis is aligned with the velocity vector p_d, i.e., $x_w := \dot{p}_d / \|\dot{p}_d\|$.
  - The acceleration can be written as

    $$\ddot{p}_d = g + f_I/m,$$

    where f_I is the aerodynamic/propulsive force in inertial frame, and g is the gravity vector.
  - The main sources of forces for an airplane are the engine thrust and lift. Both are approximately contained within the symmetry plane of the aircraft. Hence, the z_w axis is chosen such that the (x_w, z_w) plane contains f_I.
  - The y_w axis is chosen to complete a right-handed orthonormal triad.

- Assume that we can control independently:
  - The tangential acceleration[1] a_t along the wind velocity vector (x_w axis).
  - The normal acceleration a_n (along the z_w axis).
  - The roll rate ω_1 around the wind velocity vector (x_w axis).
- Recall $\ddot{p}_d = g + a_I = g + Ra_w$, where a_I is the acceleration in the inertial frame, a_w is the acceleration in the wind frame defined in the previous slide, and R is a rotation matrix, computed as a function of $\dot{p}_d$, $\ddot{p}_d$.
- Differentiating,

$$p_d^{(3)} = \dot{R}a_w + R\dot{a}_w = R(\omega \times a_w) + R\dot{a}_w.$$

- In addition, since $\dot{p}_d = VRe_1$ (coordinated flight), we also have

$$\ddot{p}_d = g + Ra_w = \dot{V}Re_1 + VR(\omega \times e_1)$$

[1] Here and below acceleration is understood as "acceleration due to aerodynamic/propulsive forces."

- In coordinates, the second equation reads:

$$\ddot{p}_d = R \begin{bmatrix} \dot{V} \\ V\omega_3 \\ -V\omega_2 \end{bmatrix},$$

i.e., ω_2 and ω_3 can be computed from $\dot{p}$ and $\ddot{p}$.

- Also,

$$\frac{d}{dt^3} \begin{bmatrix} p_{d,1} \\ p_{d,2} \\ p_{d,3} \end{bmatrix} = R \begin{bmatrix} \omega_2 a_n + \dot{a}_t \\ \omega_3 a_t - \omega_1 a_n \\ -\omega_2 a_t + \dot{a}_n \end{bmatrix}$$

- Finally, we have

$$\begin{bmatrix} \dot{a}_t \\ \omega_1 \\ \dot{a}_n \end{bmatrix} = \begin{bmatrix} -\omega_2 a_n \\ \omega_3 a_t / a_n \\ \omega_2 a_t \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/a_n & 0 \\ 0 & 0 & 1 \end{bmatrix} R^T \frac{d^3}{dt^3} \begin{bmatrix} p_{d,1} \\ p_{d,2} \\ p_{d,3} \end{bmatrix}$$

- I.e., the system is differentially flat, with flat output p_d, as long as
  - $V = \|\dot{p}_d\| \neq 0$: if the velocity is zero, then coordinated flight is not well defined.
  - $a_n = \left(I - \frac{1}{V^2}\dot{p}_d \dot{p}_d^T\right)(\ddot{p}_d - g) \neq 0$: if the normal acceleration is zero, the roll attitude is not well defined.

Incorporating aerodynamics and propulsive models

- We have derived a differentially flat system for aircraft dynamics, with flat output p_d (position trajectory) and inputs: $(\dot{a}_t, \omega_1, \dot{a}_n)$.
- How can we control a_t, a_n (or their derivatives)?
  - The wing lift is
    $$L = \frac{1}{2m}\rho V^2 S C_{L_\alpha}\alpha + a_{L_0},$$
    the drag is similarly computed.
  - The engine thrust T is a function of the throttle setting δ_T and other variables.
  - The acceleration components in wind axes are given by
    $$a_t = T(\delta_T)\cos\alpha - D(\alpha),$$
    $$a_n = -T(\delta_T)\sin\alpha - L(\alpha).$$
- Compute, e.g., α and δ_T, from the desired a_t, a_n.
- Rely on a CAS that tracks the commanded α, δ_T, and ω_1.

Adding feedback

- So far, we have shown that, given a reference trajectory, we can compute uniquely (modulo a 180° roll rotation) the corresponding state and control input trajectories.
- What if the initial condition is not on the reference trajectory? What if there are disturbances that make the aircraft deviate from the trajectory? We need feedback.
- Let $p : t \mapsto p(t)$ be the actual position of the vehicle, and consider a system in which $p^{(3)} = u$, e.g.,

$$\frac{d}{dt}\begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \dot{p} \\ \ddot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} u$$

Tracking control law

- Define the error $e = p - p_d$. The error dynamics are given by

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}\left(u - p_d^{(3)}\right).$$

- If we
  - set
  $$u = p_d^{(3)} - K[e, \dot{e}, \ddot{e}]^T,$$
  where k is a stabilizing control gain for the error dynamics, and
  - compute a_t, a_n, ω_1 from $(p, \dot{p}, \ddot{p}, u)$ (vs. p_d and its derivatives)
- then,

$$\lim_{t\to+\infty} e(t) = \lim_{t\to+\infty}(p(t) - p_d(t)) = 0,$$

as desired.

Some remarks

- Convergence assured "almost" globally, the control law breaks down if at any point $\dot{x} = 0$, or $a_n = 0$.
- A modification of this control law can ensure path tracking (vs. trajectory tracking), requiring less thrust control effort. See Hauser & Hindman '97.
- Some limitations:
  - Simplified aerodynamic/propulsive models.
  - Saturations are not taken into account (unbounded a_n, a_t, ω_1).
  - Coordinated flight is an additional constraints, no control over roll: this model cannot account for, e.g., a split-S maneuver.