Topic #22

16.30/31 Feedback Control Systems

Analysis of Nonlinear Systems

- Lyapunov Stability Analysis

Lyapunov Stability Analysis

- Very general method to prove (or disprove) stability of nonlinear systems.
  - Formalizes idea that all systems will tend to a "minimum-energy" state.
  - Lyapunov's stability theory is the **single most powerful method in stability analysis of nonlinear systems**.

- Consider a nonlinear system $\dot{\mathbf{x}} = f(\mathbf{x})$
  - A point $\mathbf{x}_0$ is an equilibrium point if $f(\mathbf{x}_0) = 0$
  - Can always assume $\mathbf{x}_0 = 0$

- In general, an equilibrium point is said to be
  - **Stable in the sense of Lyapunov** if (arbitrarily) small deviations from the equilibrium result in trajectories that stay (arbitrarily) close to the equilibrium for all time.
  - **Asymptotically stable** if small deviations from the equilibrium are eventually "forgotten," and the system returns asymptotically to the equilibrium point.
  - **Exponentially stable** if it is asymptotically stable, and the convergence to the equilibrium point is "fast."

Stability

- Let $\mathbf{x} = 0 \in D$ be an equilibrium point of the system

$$\dot{\mathbf{x}} = f(\mathbf{x}),$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}$

  - $f(\mathbf{x})$ is locally Lipschitz in D if $\forall \mathbf{x} \in D\ \exists I(\mathbf{x})$ such that $|f(\mathbf{y}) - f(\mathbf{z})| \leq L|\mathbf{y} - \mathbf{z}|$ for all $\mathbf{y}$, $\mathbf{z} \in I(\mathbf{x})$.
  - Smoothness condition for functions which is stronger than regular continuity – intuitively, a Lipschitz continuous function is limited in how fast it can change. (see here)
  - A sufficient condition for a function to be Lipschitz is that the Jacobian $\partial f / \partial \mathbf{x}$ is uniformly bounded for all $\mathbf{x}$.

- The equilibrium point is
  - **Stable in the sense of Lyapunov (ISL)** if, for each $\varepsilon \geq 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| \leq \varepsilon, \quad \forall t \geq 0;$$

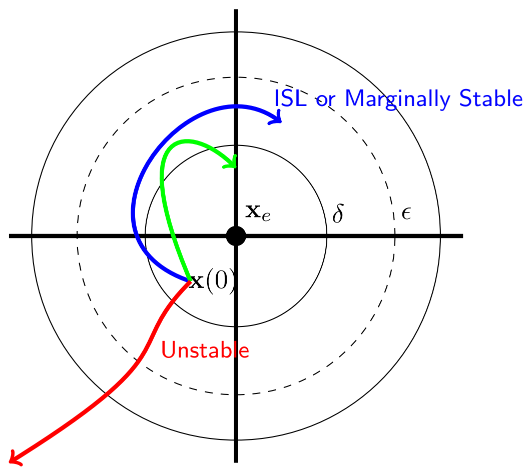
  - **Asymptotically stable** if stable, and there exists $\delta > 0$ s.t.

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \lim_{t \to +\infty} \mathbf{x}(t) = 0$$

  - **Exponentially stable** if there exist $\delta, \alpha, \beta > 0$ s.t.

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \beta e^{-\alpha t}, \forall t \geq 0;$$

  - **Unstable** if not stable.



- How do we analyze the stability of an equilibrium point?

- Already talked about how to linearize the dynamics about the equilibrium point and use the conclusion from the linear analysis to develop a **local** conclusion
  - Often called *Lyapunov's first method*

- How about a more global conclusion?
  - Powerful method based on concept of **Lyapunov function**
    - *Lyapunov's second method*
  - LF is a scalar function of the state that is always non-negative, is zero only at the equilibrium point, and is such that its value is non-increasing along system's trajectories.

- Generalization of result from classical mechanics, which is that a vibratory system is stable if the total energy is continually decreasing.

Lyapunov Stability Theorem

- Let D be a compact subset[1] of the state space, containing the equilibrium point (i.e., $\{\mathbf{x}_0\} \subset D \subset \mathbb{R}^n$), and a let there be a function $V : D \to \mathbb{R}$.

- **Theorem:** The equilibrium point $\mathbf{x}_0$ is stable (in the sense of Lyapunov) if the V satisfies the following conditions (and if it does, it is called a Lyapunov function):

  1. $V(\mathbf{x}) \geq 0$, for all $\mathbf{x} \in D$.

  2. $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{x}_0$.

  3. For all $\mathbf{x}(t) \in D$,

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) \equiv \frac{d}{dt}V(\mathbf{x}(t)) &= \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}(t)}{dt} \\ &= \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \cdot f(\mathbf{x}) \leq 0 \end{aligned}$$

- Furthermore,

  1. If $\dot{V}(\mathbf{x}(t)) = 0$ only when $\mathbf{x}(t) = \mathbf{x}_0$, then the equilibrium is **asymptotically stable**.

  2. If $\dot{V}(\mathbf{x}(t)) < -\alpha V(\mathbf{x}(t))$, for some $\alpha > 0$, then the equilibrium is **exponentially stable**.

- Finally, to ensure **global stability**, need to impose extra condition that as $\|\mathbf{x}\| \to +\infty$, then $V(\mathbf{x}) \to +\infty$.

  - Such a function V is said **radially unbounded**

[1] A compact set is a set that is closed and bounded, e.g., the set $\{(x, y) : 0 \leq x \leq 1, -x^2 \leq y \leq x^2$.

- Note that condition (1) in the Theorem corresponds to $V(\mathbf{x})$ being positive definite ($V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq 0$ and $V(0) = 0$.)

  $V(\mathbf{x})$ being positive semi-definite means $V(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, but $V(\mathbf{x})$ can be zero at points other than $\mathbf{x} = 0$.)

  i) $V(\mathbf{x}) = x_1^2 + x_2^2$ PD, PSD, ND, NSD, ID

  ia) $V(\mathbf{x}) = x_1^2$ PD, PSD, ND, NSD, ID

  ii) $V(\mathbf{x}) = (x_1 + x_2)^2$ PD, PSD, ND, NSD, ID

  iii) $V(\mathbf{x}) = -x_1^2 - (3x_1 + 2x_2)^2$ PD, PSD, ND, NSD, ID

  iv) $V(\mathbf{x}) = x_1 x_2 + x_2^2$ PD, PSD, ND, NSD, ID

  v) $V(\mathbf{x}) = x_1^2 + \frac{2x_2^2}{1+x_2^2}$ PD, PSD, ND, NSD, ID

Example 1: Pendulum

- Typical method for finding candidate Lyapunov functions is based on the mechanical energy in the system

- Consider a pendulum:

$$\ddot{\theta} = -\frac{g}{l}\sin(\theta) - c\dot{\theta},$$

  - Setting $x_1 = \theta$, $x_2 = \dot{\theta}$:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin(x_1) - cx_2\end{aligned}$$

- Can use the mechanical energy as a Lyapunov function candidate:

$$V = \frac{1}{2}ml^2x_2^2 + mgl(1 - \cos(x_1))$$

- Analysis:

$$\begin{aligned}V(0) &= 0 \\ V(x_1, x_2) &\geq 0\end{aligned}$$

$$\begin{aligned}\dot{V}(x_1, x_2) &= (ml^2x_2)\dot{x}_2 + mgl\sin(x_1)\dot{x}_1 \\ &= -cml^2x_2^2 \leq 0\end{aligned}$$

- Thus the equilibrium point $(x_1, x_2) = 0$ is stable in the sense of Lyapunov.
  - But note that $\dot{V}$ is only NSD

Example 2: Linear System

- Consider a system $\dot{\mathbf{x}} = A\mathbf{x}$.

- Another common choice: quadratic Lyapunov functions,
$$V(\mathbf{x}) = \|M\mathbf{x}\|^2 = \mathbf{x}^T M^T M\mathbf{x} = \mathbf{x}^T P\mathbf{x}$$
with $P = M^T M$, a symmetric and positive definite matrix.
  - Easy to check that $V(0) = 0$, and $V(\mathbf{x}) \geq 0$

- To find the derivative along trajectories, note that
$$\begin{aligned} \dot{V}(\mathbf{x}) &= \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} \\ &= \mathbf{x}^T A^T P\mathbf{x} + \mathbf{x}^T PA\mathbf{x} \\ &= \mathbf{x}^T (A^T P + PA)\mathbf{x} \end{aligned}$$

- Next step: make this derivative equal to a given negative-definite function
$$\dot{V}(\mathbf{x}) = \mathbf{x}^T (A^T P + PA)\mathbf{x} = -\mathbf{x}^T Q\mathbf{x}, \quad (Q > 0)$$

- Then appropriate matrix P can be found by solving:
$$A^T P + PA = -Q$$
  - Not surprisingly, this is called a **Lyapunov equation**
  - Note that it happens to be the linear part of a Riccati equation
  - It always has a solution if all the eigenvalues of A are in the left half plane (i.e., A is Hurwitz, and defines a stable linear system)

Example 3: Controlled Linear System

- Consider a possibly unstable, but controllable linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

- We know that if we solve the Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

and set $\mathbf{u} = K\mathbf{x}$ with $K = -R^{-1}B^T P$, the closed-loop system is stable.

$$\dot{\mathbf{x}} = (A + BK)\mathbf{x}$$

  - Can confirm this fact using the Lyapunov Thm.

- In particular, note that the solution P of the Riccati equation has the interpretation of a Lyapunov function, i.e., for this closed-loop system we can use

$$V(\mathbf{x}) = \mathbf{x}^T P\mathbf{x}$$

  - Check:

$$\begin{aligned}
\dot{V}(\mathbf{x}) &= \mathbf{x}^T P\dot{\mathbf{x}} + \dot{\mathbf{x}}^T P\mathbf{x} \\
&= \mathbf{x}^T P(A + BK)\mathbf{x} + \mathbf{x}^T(A + BK)^T P\mathbf{x} \\
&= \mathbf{x}^T(PA + PBK + A^T P + K^T B^T P)\mathbf{x} \\
&= \mathbf{x}^T(A^T P + PA - PBR^{-1}B^T P - PBR^{-1}B^T P)\mathbf{x} \\
&= -\mathbf{x}^T(Q + PBR^{-1}B^T P)\mathbf{x} \leq 0
\end{aligned}$$

Example 4: Local Region

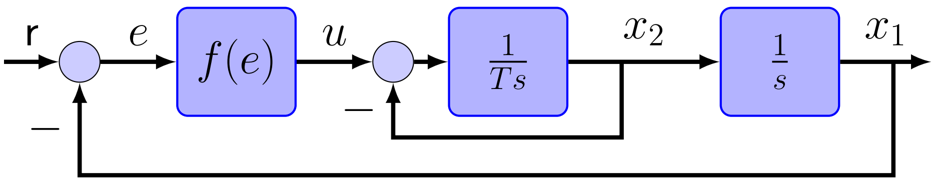
- Consider the system
$$\frac{dx}{dt} = \frac{2}{1+x} - x$$
which has equilibrium points at $x = 1$ and $x = -2$.

- Around the eq point $x = 1$, let $z = x - 1$, then
$$\frac{dz}{dt} = \frac{2}{2+z} - z - 1$$
which has an eq point at $z = 0$.
  - Consider LF $V = \frac{1}{2}z^2$ which is global PD
  - Then can show
$$\dot{V} = z\dot{z} = \frac{2z}{2+z} - z^2 - z$$

- Now restrict attention to an interval B_r, where $r < 2$ and thus $z < 2$ and $-2 < z$, which can be rewritten as $2 + z > 0$, then have
$$\begin{aligned} \dot{V}(2+z) &= 2z - (z^2 + z)(2+z) \\ &= -z^3 - 3z^2 \\ &= -z^2(z+3) < 0 \;\forall z \in B_r (r < 2) \end{aligned}$$

- Thus it follows that $\dot{V} < 0$ for all $z \in B_r$, $z \neq 0$ and hence the eq point $x_e = 1$ is locally asymptotically stable.

Example 5: Saturation



- System dynamics are

$$\begin{aligned}\dot{e} &= -x_2 \\ \dot{x}_2 &= -\frac{1}{T}x_2 + \frac{f(e)}{T}\end{aligned}$$

where it is known that:

  - $u = f(e)$ and $f(\cdot)$ lies in the first and third quadrants
  - $f(e) = 0$ means $e = 0$, and $\int_0^e f(e)de > 0$
  - Assume that $T > 0$ so open loop stable

- Candidate Lyapunov function

$$V = \frac{T}{2}x_2^2 + \int_0^e f(\sigma)d\sigma$$

- Clearly:
  - $V = 0$ if $e = x_2 = 0$ and $V > 0$ for $x_2^2 + e^2 \neq 0$
  - What about the derivative?

$$\begin{aligned}\dot{V} &= Tx_2\dot{x}_2 + f(e)\dot{e} \\ &= Tx_2\left[-\frac{1}{T}x_2 + \frac{f(e)}{T}\right] + f(e)\left[-x_2\right] \\ &= -x_2^2\end{aligned}$$

- Since V PD and $\dot{V}$ NSD, the origin is stable ISL.

Invariance Principle

- Lyapunov's theorem ensures asymptotic stability if we can find a Lyapunov function that is strictly decreasing away from the equilibrium.
  - Unfortunately, in many cases (e.g., in aerospace, robotics, etc.), there may be situations in which $\dot{V} = 0$ for states other than at the equilibrium. (i.e. $\dot{V}$ is NSD not ND)
  - Need further analysis tool for these types of systems, since stable ISL is typically insufficient

- **LaSalle's invariance principle** Consider a system

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

  - Let $\Omega \in D$ be a (compact) positively invariant set, i.e., a set such that if $\mathbf{x}(t_0) \in \Omega$, then $\mathbf{x}(t) \in \Omega$ for all $t \geq t_0$.
  - Let $V : D \to \mathbb{R}$, such that $\dot{V}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega$.

  Then, $\mathbf{x}(t)$ will eventually approach the largest positively invariant set in which $\dot{V} = 0$.

- Note that positively invariant sets include equilibrium points and limit cycles.

Invariance Example 1

- Pendulum Revisited – consider again the mechanical energy as the Lyapunov function
  - Showed that $\dot{V}(\mathbf{x}) = -cml^2 x_2^2 \sim \dot{\theta}^2$
  - Thus previously could only show that $\dot{V}(\mathbf{x}) \leq 0$, and the system is stable ISL
  - But we know that $\dot{V}(\mathbf{x}) = 0$ whenever $\dot{\theta} = 0$, i.e., the system is on the $x_2 = \dot{\theta} = 0$ axis
  - However, the only part of the $\mathbf{x}_2 = 0$ axis that is invariant is the origin!
  - LaSalle's invariance principle allows us to conclude that the pendulum system response must tend to this invariant set
  - Hence the system is in fact asymptotically stable.

- Revisit Example 5:
  - $\dot{V}$ decreasing if $x_2 \neq 0$, and the only invariant point is $x_2 = e = 0$, so the origin is asymptotically stable

Invariance Example 2

- Limit cycle:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1^7[x_1^4 + 2x_2^2 - 10]\\ \dot{x}_2 &= -x_1^3 - 3x_2^5[x_1^4 + 2x_2^2 - 10]\end{aligned}$$

  - Note that $x_1^4 + 2x_2^2 - 10$ is invariant since

  $$\frac{d}{dt}[x_1^4 + 2x_2^2 - 10] = -(4x_1^{10} + 12x_2^6)(x_1^4 + 2x_2^2 - 10)$$

  which is zero if $x_1^4 + 2x_2^2 = 10$.

  - Dynamics on this set governed by $\dot{x}_1 = x_2$ and $\dot{x}_2 = -x_1^3$, which corresponds to a limit cycle with clockwise state motion in the phase plane

- Is the limit cycle attractive? To determine, pick

$$V = (x_1^4 + 2x_2^2 - 10)^2$$

which is a measure of the distance to the LC.

  - In a region about the LC, can show that

  $$\dot{V} = -8(x_1^{10} + 3x_2^6)(x_1^4 + 2x_2^2 - 10)^2$$

  so $\dot{V} < 0$ except if $x_1^4 + 2x_2^2 = 10$ (the LC) or $x_1^{10} + 3x_2^6 = 0$ (at origin).

- Conclusion: since the origin and LC are the invariant set for this system - thus all trajectories starting in a neighborhood of the LC converge to this invariant set

  - Actually turns out the origin in unstable.

Summary

- Lyapunov functions are a very powerful tool to study stability of a system.

- Lyapunov's theorem only gives us a sufficient condition for stability
  - If we can find a Lyapunov function, then we know the equilibrium is stable.
  - However, if a candidate Lyapunov function does not satisfy the conditions in the theorem, **this does not prove** that the equilibrium is unstable.

- Unfortunately, there is no general way for constructing Lyapunov functions; however,
  - Often energy can be used as a Lyapunov function.
  - Quadratic Lyapunov functions are commonly used; these can be derived from linearization of the system near equilibrium points.

  - A very recent development: "Sum-of-squares" methods can be used to construct polynomial Lyapunov functions.

- LaSalle's invariance principle very useful in resolving cases when $\dot{V}$ is negative semi-definite.