Topic #21

16.30/31 Feedback Control Systems

Systems with Nonlinear Functions

- Describing Function Analysis

NL Example

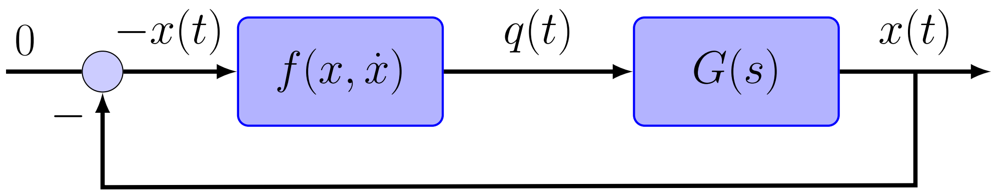
- Another classic example – Van Der Pol equation[1]:

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0$$

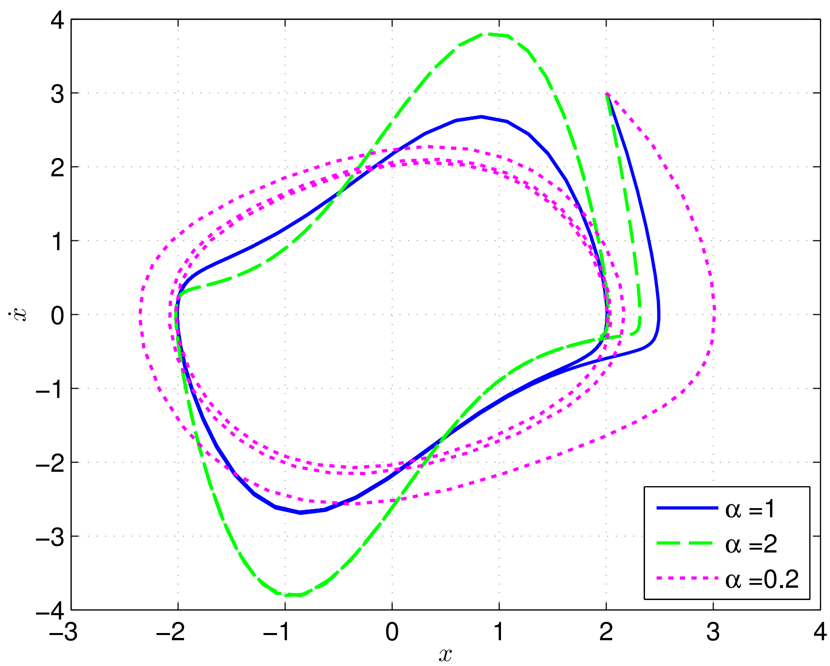
which can be written as linear system

$$G(s) = \frac{\alpha}{s^2 - \alpha s + 1}$$

in negative feedback with a nonlinear function $f(x, \dot{x}) = x^2\dot{x}$



  - Would expect to see different behaviors from the system depending on the value of α



- Of particular concern is the existence of a **limit cycle** response
  - **Sustained oscillation** for a nonlinear system, of the type above

[1] Slotine and Li, page 158

- In this case the signal $x(t)$ would be of the form of an oscillation $x(t) = A\sin(\omega t)$ so that $\dot{x}(t) = A\omega\cos(\omega t)$
    - Note that A and ω are not known, and may not actually exist.

- Given the form of $x(t)$, we have that

$$\begin{aligned} q(t) = -x^2\dot{x} &= -A^2\sin^2(\omega t)A\omega\cos(\omega t) \\ &= -\frac{A^3\omega}{4}(\cos(\omega t) - \cos(3\omega t)) \end{aligned}$$

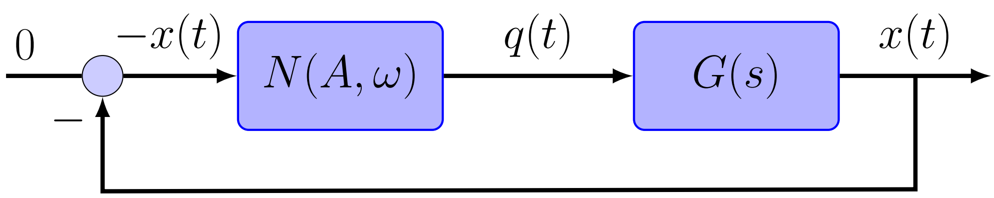
- Thus the output of the nonlinearity (input of the linear part) contains the third harmonic of the input
    - **Key point:** since the system $G(s)$ is low pass, expect that this third harmonic will be "sufficiently attenuated" by the linear system that we can approximate

$$\begin{aligned} q(t) = -x^2\dot{x} &\approx -\frac{A^3\omega}{4}\cos(\omega t) \\ &= \frac{A^2}{4}\frac{d}{dt}[-A\sin(\omega t)] \end{aligned}$$

- Note that we can now create an **effective "transfer function"** of this nonlinearity by defining that:

$$q = N(A,\omega)(-x) \quad \Rightarrow \quad N(A,\omega) = \frac{A^2 j\omega}{4}$$

which approximates the effect of the nonlinearity as a frequency response function.



- What are the implications of adding this nonlinearity into the feedback loop?
  - Can approximately answer that question by looking at the stability of $G(s)$ in feedback with N.

    $$x = A\sin(\omega t) = G(j\omega)q = G(j\omega)N(A,\omega)(-x)$$

    which is equivalent to:

    $$(1 + G(j\omega)N(A,\omega))x = 0$$

    that we can rewrite as:

    $$1 + \frac{A^2(j\omega)}{4}\,\frac{\alpha}{(j\omega)^2 - \alpha(j\omega) + 1} = 0$$

    which is only true if $A = 2$ and $\omega = 1$

- These results suggest that we could get sustained oscillations in this case (i.e. a limit cycle) of amplitude 2 and frequency 1.
  - This is consistent with the response seen in the plots - independent of α we get sustained oscillations in which the $x(t)$ value settles down to an amplitude of 2.
  - Note that α does impact the response and changes the shape/features in the response.

- Approach (called **Describing Functions**) is generalizable....

Describing Function Analysis

- Now consider a more general analysis of the describing function approach.

- In this case consider the input to the nonlinearity to be $x(t) = A \sin \omega t$.
  - Would expect that the output $y = f(x)$ is a complex waveform, which we represent using a Fourier series of the form:

$$y(t) = b_0 + \sum_{n=1}^{\infty} (a_n \sin n\omega t + b_n \cos n\omega t)$$

- So it is explicit that the output of the nonlinearity contains multiple harmonics of the ingoing signal.
  - In general we would expect these harmonics to pass through the system $G(s)$ and show up in the input to the nonlinearity
  - Would then have a much more complicated input for $x(t)$, leading to a more complex output $y(t) \Rightarrow$ non-feasible analysis path

- Need approximate approach, so assume
  - The **fundamental** $y_f = a_1 \sin \omega t + b_1 \cos \omega t$ is significantly larger in amplitude than the harmonics
  - The linear system $G(s)$ acts as a low-pass that attenuates the harmonics more strongly than the fundamental.

- As a result, can approximate $y(t)$ as y_f, and then the **describing function** of the nonlinearity becomes

$$N = \frac{y_f}{x}$$

- Using Fourier analysis, can show that

$$a_1 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} y(t)\sin\omega t \; dt \qquad b_1 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} y(t)\cos\omega t \; dt$$

  - Note that will often find that N is a function of the amplitude A and the frequency ω.

- Simple example: ideal relay $y = T$ if $x \geq 0$, otherwise $y = -T$. Then (setting $\omega = 1$ for simplicity, since the solution isn't a function of ω)

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin t \; dt = \frac{2}{\pi}\int_0^{\pi} T\sin t \; dt = \frac{4T}{\pi}$$

  - Nonlinearity is odd (i.e., $y(-t) = -y(t)$), so $b_i = 0 \; \forall \; i$

- So we have

$$N = \frac{4T}{\pi A}$$

so the equivalent gain decreases as the input amplitude goes up.

  - Makes sense since the output is limited, so effective gain of the nonlinearity must decrease as the amplitude of the input goes up

Saturation Nonlinearity

- Classic nonlinearity is the saturation function

$$u = f(e) = \begin{cases} T & \text{if } e > T \\ e & \text{if } -T \leq e \leq T \\ -T & \text{if } e < -T \end{cases}$$

  - Outputs the signal, but only up to some limited magnitude, then caps the output to a value T.
  - Saturation is an odd function

- Describing function calculation is (as before $b_i = 0$):
  - Assume $e(t) = A \sin \omega t$ and $A > T$, and find ψ_T so that

$$e(t_T) = A \sin \psi_T = T \quad \Rightarrow \quad \psi_T = \arcsin\left(\frac{T}{A}\right)$$

- Set $\psi = \omega t$, so that $d\psi = \omega dt$

$$\begin{aligned} a_1 &= \frac{4\omega}{\pi} \int_0^{\pi/2} y(t) \sin \psi \, \frac{d\psi}{\omega} \\ &= \frac{4}{\pi} \int_0^{\psi_T} A \sin \psi \sin \psi \, d\psi + \frac{4}{\pi} \int_{\psi_T}^{\pi/2} T \sin \psi \, d\psi \\ &= \frac{4A}{\pi} \int_0^{\psi_T} A \sin^2 \psi \, d\psi + \frac{4T}{\pi} \int_{\psi_T}^{\pi/2} \sin \psi \, d\psi \\ &= \frac{2A}{\pi} \arcsin\left(\frac{T}{A}\right) + \frac{2T}{\pi} \sqrt{1 - \left(\frac{T}{A}\right)^2} \end{aligned}$$

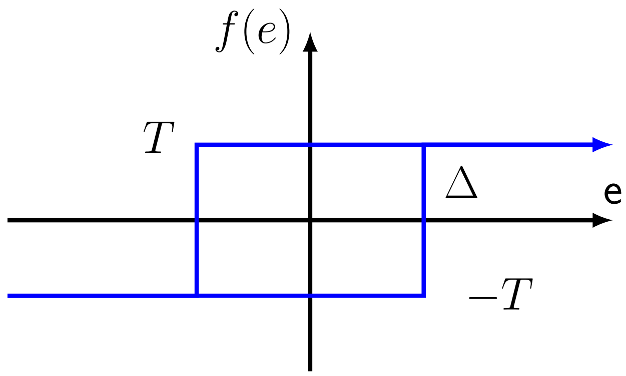
- So if $A > T$ the DF is given by

$$N(A) = \frac{2}{\pi} \left[\arcsin\left(\frac{T}{A}\right) + \left(\frac{T}{A}\right) \sqrt{1 - \left(\frac{T}{A}\right)^2} \right]$$

and if $A < T$, then $N(A) = 1$.

Odd Nonlinearities with Memory

- Many of the DF are real, but for NL with delay, hysteresis, can get very complex forms for N that involve phase lags.
  - N has both real and imaginary parts
- Example: relay with hysteresis (also known as a Schmitt trigger)



  - Converts input sine wave to square wave, and also introduces phase shift, as change from -1 to $+1$ occurs Δ after input signal has changed sign.
- If $\psi_\Delta = \arcsin(\frac{\Delta}{A})$, then

$$b_1 = \frac{2T}{\pi}\left(\int_0^{\psi_\Delta} -\cos\psi\; d\psi + \int_{\psi_\Delta}^{\pi} \cos\psi\; d\psi\right) = -\frac{4T}{\pi}\frac{\Delta}{A}$$

$$a_1 = \frac{2T}{\pi}\left(\int_0^{\psi_\Delta} -\sin\psi\; d\psi + \int_{\psi_\Delta}^{\pi} \sin\psi\; d\psi\right) = \frac{4T}{\pi}\sqrt{1-\left(\frac{\Delta}{A}\right)^2}$$

- Thus we have

$$N(A) = \frac{4T}{A\pi}\left[\sqrt{1-\left(\frac{\Delta}{A}\right)^2} - j\frac{\Delta}{A}\right]$$

  - Where the complex term arises from the phase shift from a $\sin$ input to a $\cos$ output.

Limit Cycle Analysis

- Since N is an equivalent linear gain, the stability of the loop involving both N and G is given by the condition that there be a nonzero solution to the equation

$$GN + 1 = 0 \quad \Rightarrow \quad G = \frac{-1}{N}$$

- Graphically what this will look like is an intersection between the Nyquist plot of $G(s)$ and the DF $(-1/N)$
  - If N is real, then $-1/N$ is along the negative real line

- The intersection point gives two values:
  - ω from $G(j\omega)$ at the intersection point gives the frequency of the oscillation
  - A from the expression for N for the particular value associated with the intersection point.

DF Analysis Example

- Plant: $G(s) = \frac{K}{s(T_1 s+1)(T_2 s+1)}$ with relay nonlinearity:

$$f(e) = \begin{cases} T & \text{if } e \geq 0 \\ -T & \text{if } e < 0 \end{cases}$$

- Describing function for f given by $N = 4T/(\pi A)$, and thus

$$-\frac{1}{N} = -\frac{\pi A}{4T}$$

which is on negative real line moving to left as A increases.

- Nyquist plot of $G(s)$ will cross the real line at

$$s = -\frac{K T_1 T_2}{(T_1 + T_2)}$$

with corresponding $\omega = 1/\sqrt{T_1 T_2}$

- Graphical test:

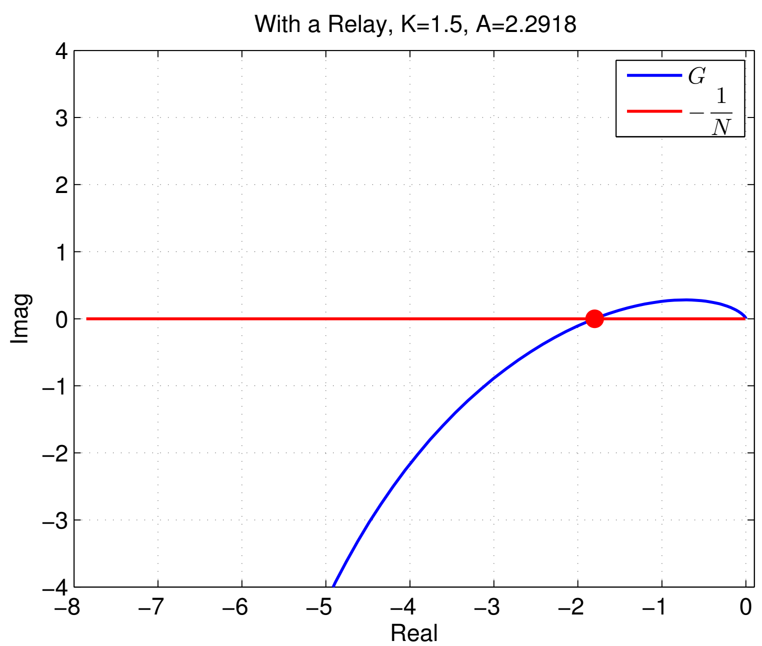


Fig. 1: DF graphical test – note that the DF is on the negative real line, parameterized by A, whereas the transfer function of G is parameterized by ω

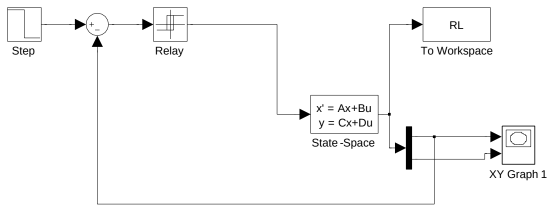


Fig. 2: Typical simulation setup

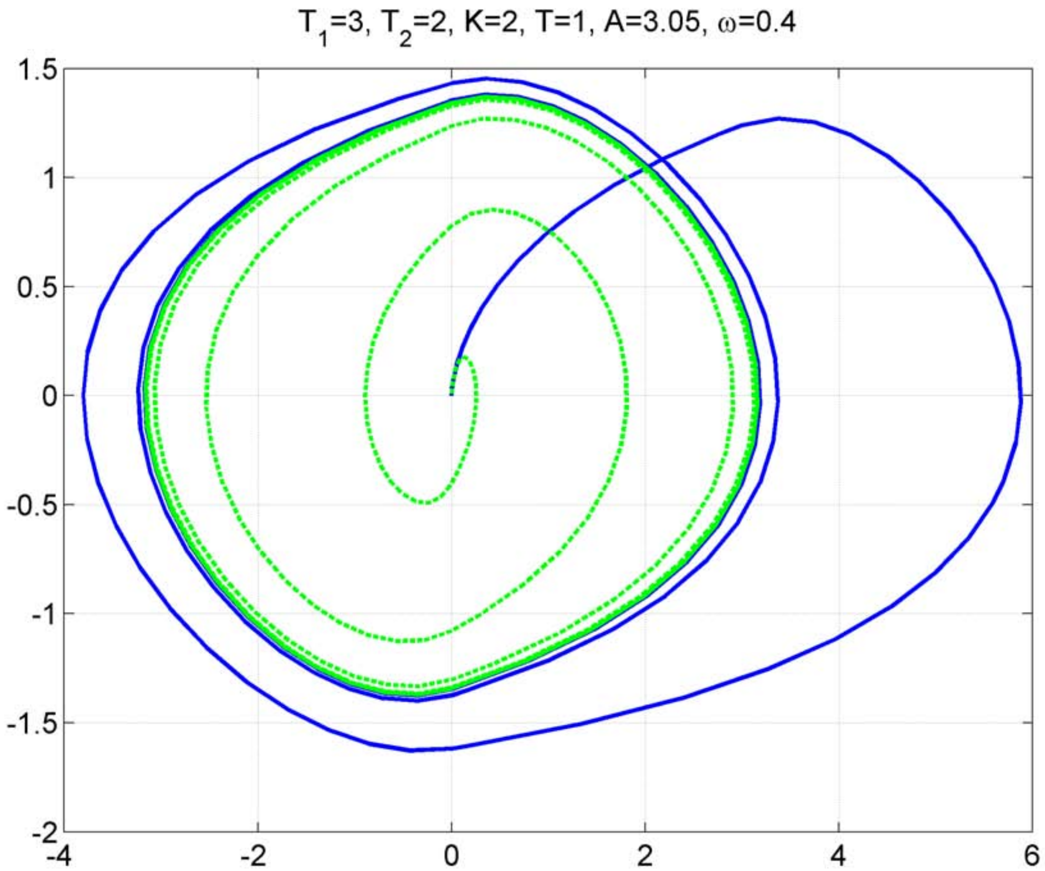


Fig. 3: System initially forced a little (green) and a lot (blue), and then both responses converge to the same limit cycle

- Can show that the expected amplitude of the limit cycle is:

$$A = \frac{4TKT_1T_2}{\pi(T_1 + T_2)}$$

- Compare with nonlinear simulation result:
  - Can we prove that this limit cycle is stable or unstable?

- Now consider the same system, but a saturation nonlinearity instead.

- For the graphical test, note that $-1/N$ is real, and very similar to the result for a relay

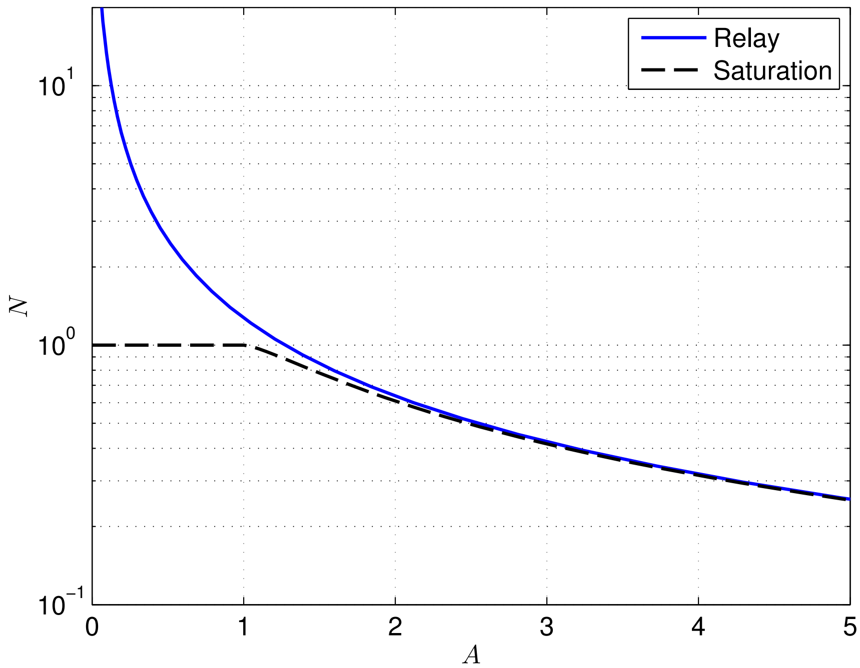
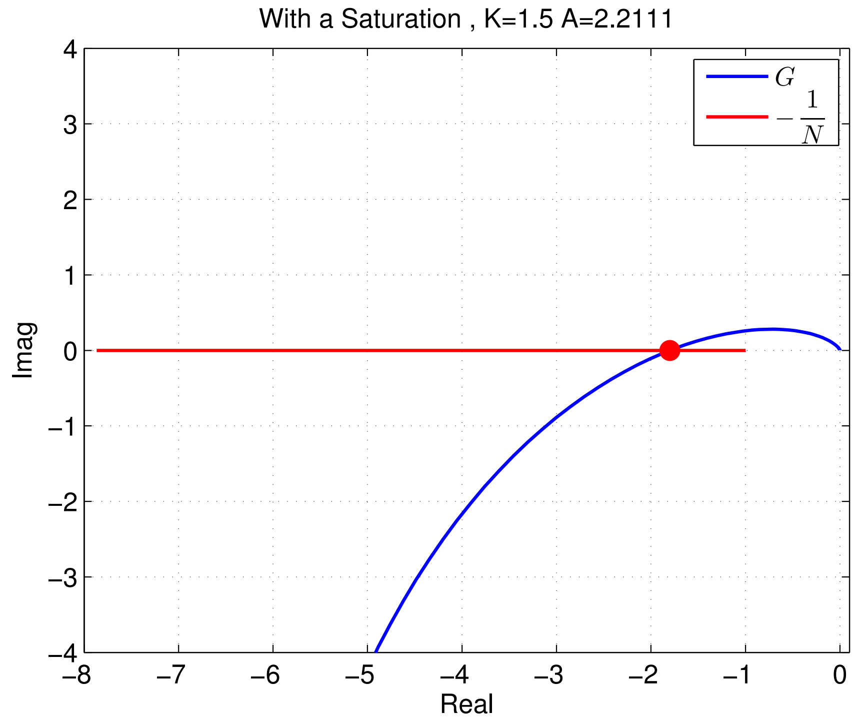


Fig. 4: Comparison of N for relay and saturation

- The slight difference being in resulting amplitude of limit cycle



- Now consider the system with a relay hysteresis with $\Delta = T/3$

- First note that if $N(A) = F - jH$, $\Rightarrow \frac{-1}{N} = \frac{F+jH}{F^2+H^2}$ and in this case

$$F^2+H^2 = \left(\frac{4T}{A\pi}\right)^2 \left(\sqrt{1-\left(\frac{\Delta}{A}\right)^2}\right)^2 + \left(\frac{4T}{A\pi}\right)^2 \left(\frac{\Delta}{A}\right)^2 = \left(\frac{4T}{A\pi}\right)^2$$

so we have

$$\frac{-1}{N} = -\left(\frac{4T}{A\pi}\right)^{-2} \left(\frac{4T}{A\pi}\right) \left[\sqrt{1-\left(\frac{\Delta}{A}\right)^2} + j\frac{\Delta}{A}\right]$$

$$= -\frac{A\pi}{4T}\left[\sqrt{1-\left(\frac{\Delta}{A}\right)^2} + j\frac{\Delta}{A}\right] = -\frac{A\pi}{4T}\sqrt{1-\left(\frac{\Delta}{A}\right)^2} - j\frac{\pi\Delta}{4T}$$

- So now use this to find when $\Im(G(j\omega_c)) = -\frac{\pi\Delta}{4T}$, and then use $\Re(G(j\omega_c))$ to find A.

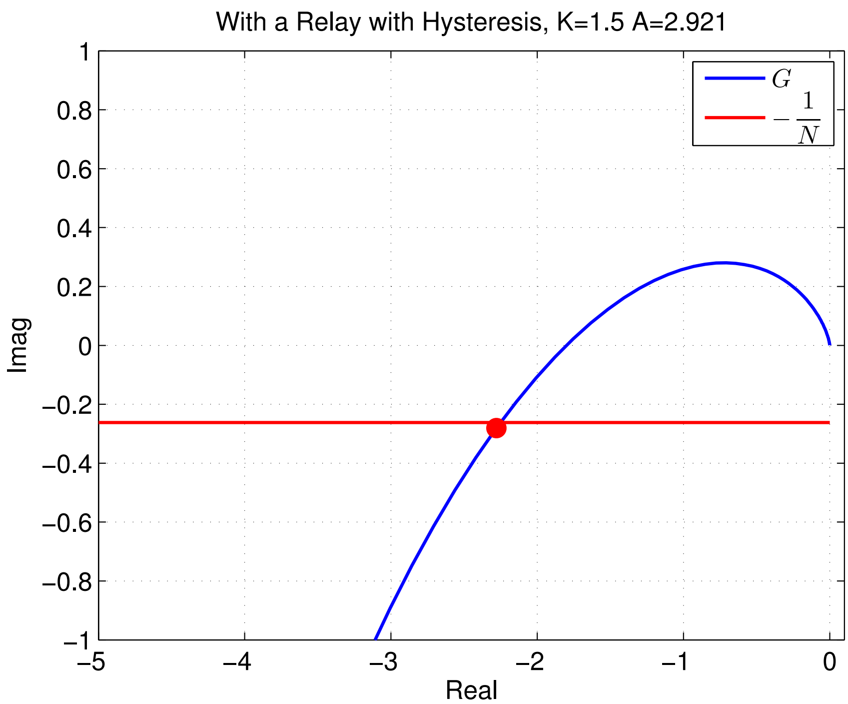


Fig. 5: Test for limit cycle for a relay with hysteresis, $K = 1.5$, $\Delta = T/3$

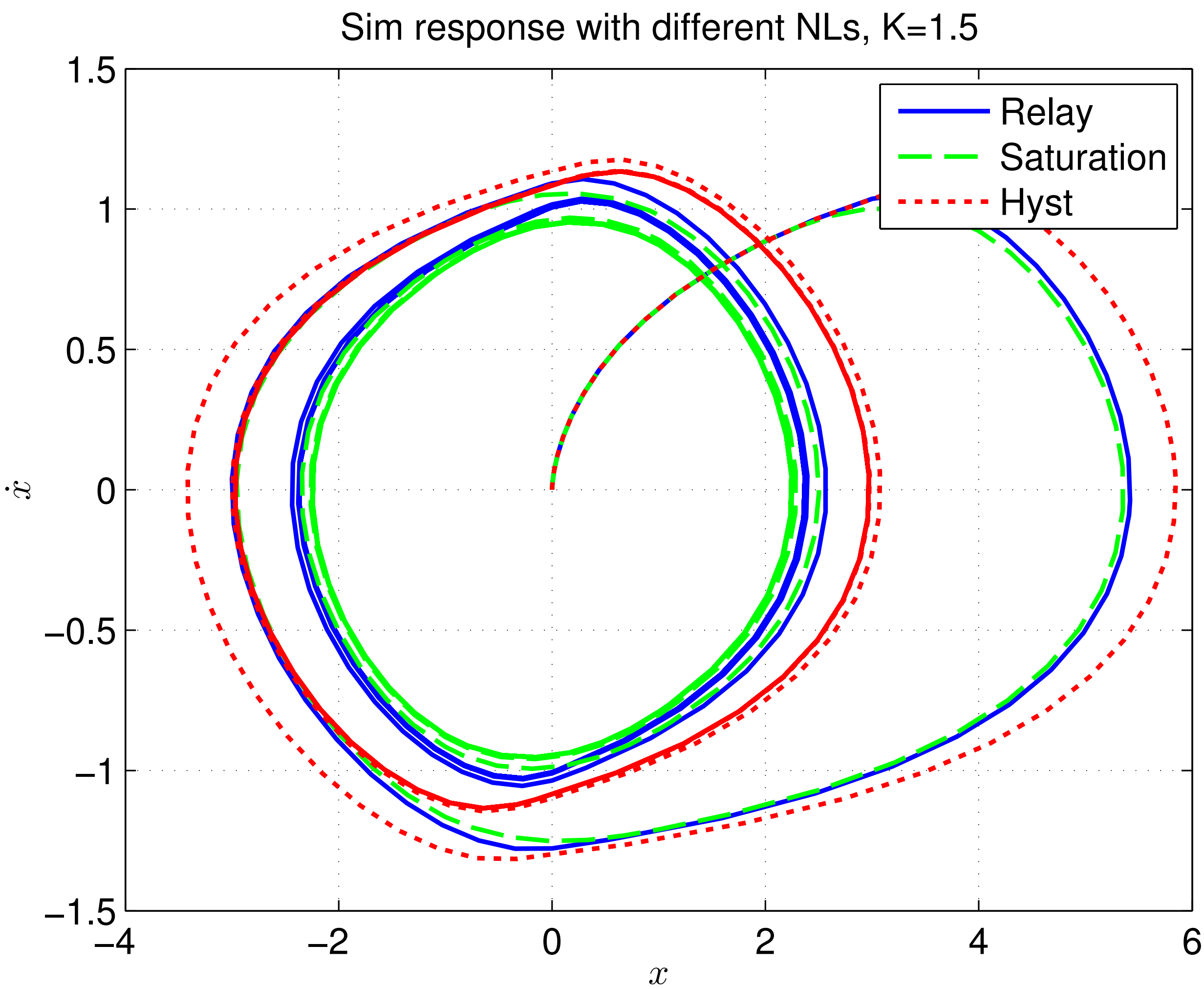


Fig. 6: Simulation comparison of all 3 types of nonlinearities with $K = 1.5$. Simulation results agree well with the predictions.

- Repeat the analysis with $K = 1$ to get the following plots
- Note greater separation in the amplitudes

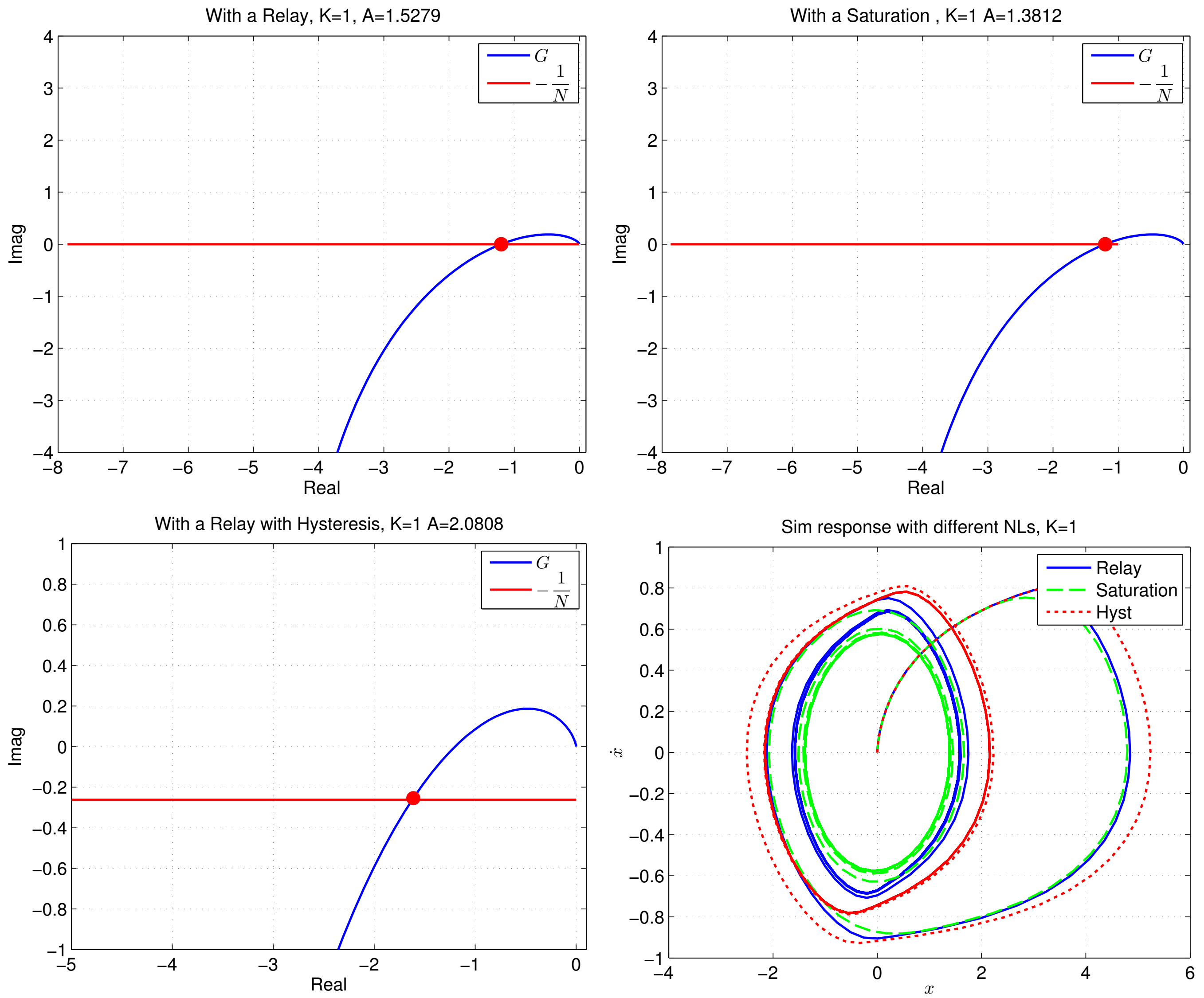


Fig. 7: Repeat all simulations with $K = 1$

- Repeat the analysis with $K = 0.7$ to get the following plots
- Now note that the saturation nonlinearity does not lead to a limit cycle

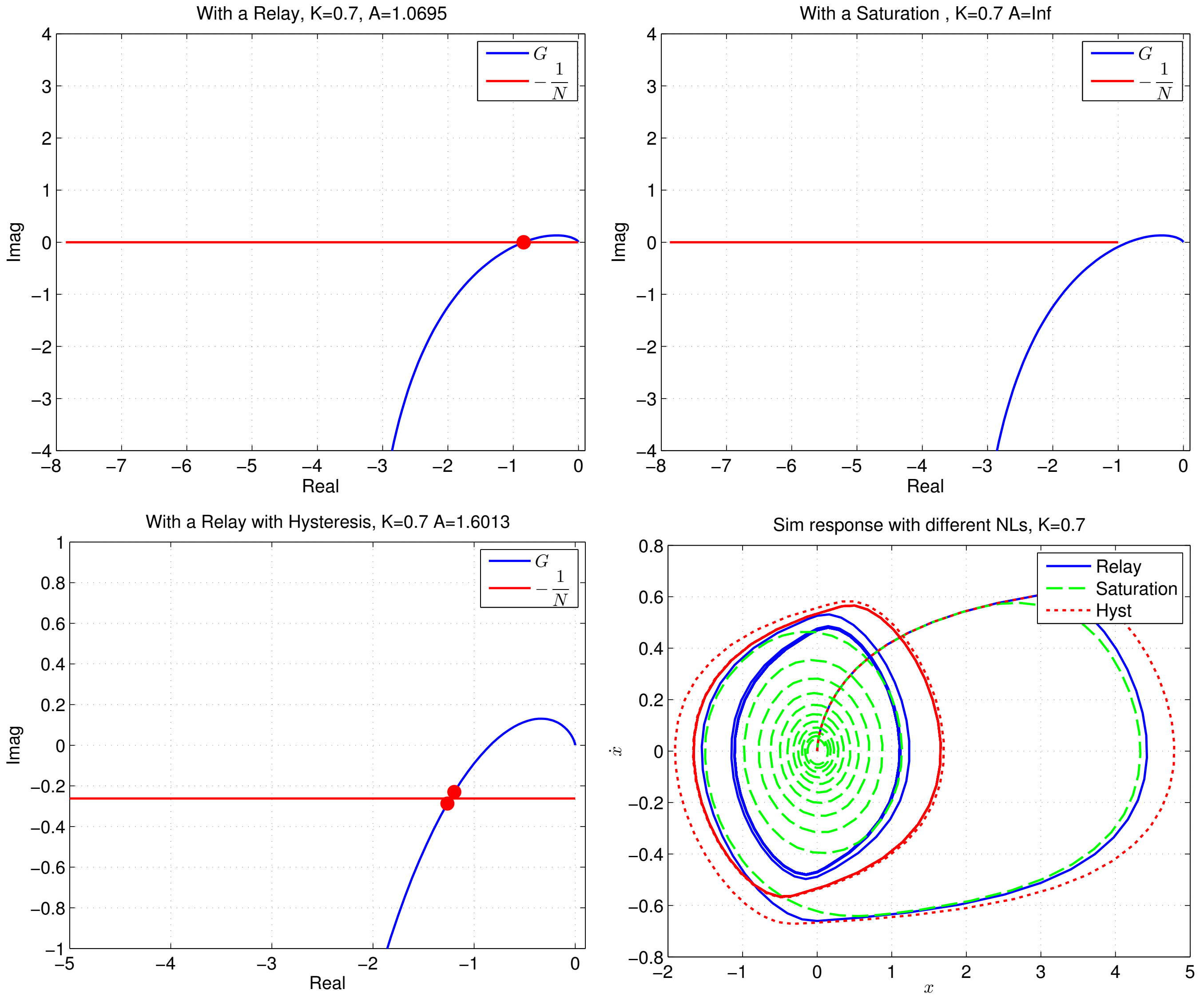


Fig. 8: Repeat all simulations with $K = 0.7$

Limit Cycle Stability

- Stability analysis is similar to that used for linear systems, where the concern is about encirclements of critical point $s = -1$.
  - Difference here: use the $-1/N(A)$ point as "critical point".

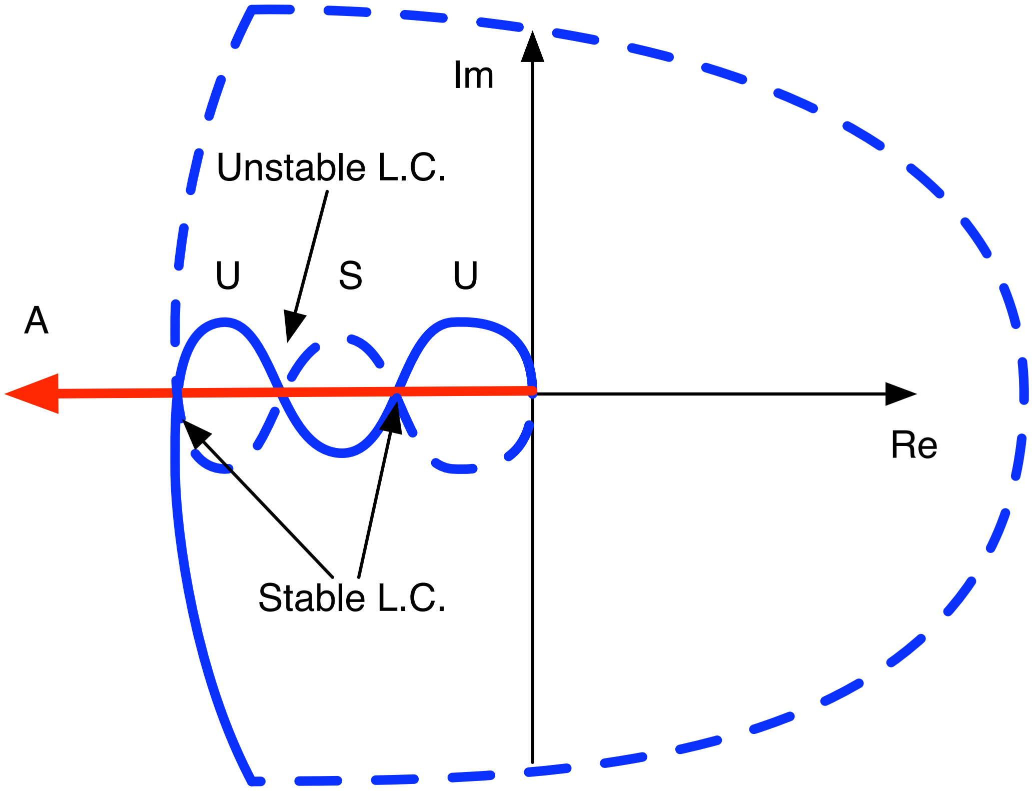
- Need to consider what the impact might be of a perturbation to amplitude A if a limit cycle is initiated.

- In cases considered, an increase in A would correspond to a shift to the left of the $-1/N(A)$ point in the s-plane
  - With that change, $G(s)$ would not encircle the critical pt, the response would be stable and the amplitude of the signal (A) would to decrease
  - Since the perturbation increase A, and response decreases it, the limit cycle is stable.

- Similarly, if a decrease in A corresponds to a shift to the right of the $-1/N(A)$ point in the s-plane
  - $G(s)$ would now encircle the critical point, the response would be unstable and the signal amplitude (A) would increase
  - Since perturbation decreases A, and response increases it, limit cycle is stable.

- So limit cycle stability hinges on $-1/N(A)$ intersecting Nyquist plot with **"increasing A pointing to the left of $G(s)$"**

Example



Code: Describing Function Analysis

```
% Examples of describing ftns
% Jonathan How
% Oct 2009
%
set(0,'DefaultLineLineWidth',1.5)
set(0,'DefaultAxesFontName','arial');set(0,'DefaultTextFontName','arial')
set(0,'DefaultlineMarkerSize',8);set(0,'DefaultlineMarkerFace','r')
set(0, 'DefaultAxesFontSize', 12);set(0, 'DefaultTextFontSize', 12);
set(0,'DefaultFigureColor','w', 'DefaultAxesColor','w',...
         'DefaultAxesXColor','k', 'DefaultAxesYColor','k',...
         'DefaultAxesZColor','k','DefaultTextColor','k')
%
%clear all
global T GG2 GG3 Delta

if 0
    T_1=3;T_2=2;K=1.5;T=1;
elseif 0
    T_1=3;T_2=2;K=1;T=1;
else
    T_1=3;T_2=2;K=0.7;T=1;
end

SS=-(K*T_1*T_2)/(T_1+T_2);
G=tf(K,conv([T_1 1 0],[T_2 1]));
omega=logspace(-3,3,400);GG=freqresp(G,omega);GG=squeeze(GG);
omega2=1/sqrt(T_1*T_2);GG2=freqresp(G,omega2);GG2=squeeze(GG2);

A1=logspace(-2,log10(10),50);
N1=4*T./(pi*A1);
figure(1);clf
plot(real(GG),imag(GG));
axis([-10 1 -10 10]);
axis([-8 .1 -4 4]);
grid on;hold on;
plot(real(-1./N1),imag(-1./N1),'r');
plot(real(GG2),imag(GG2),'ro');
hold off;
xlabel('Real');ylabel('Imag');
h=legend({'$G$','$-\frac{1}{N}$'},'Location','NorthEast','interpreter','latex');
title(['With a Relay, K=',num2str(K),', A=',num2str(-real(GG2)*4*T/pi)])

A2=logspace(log10(T),log10(10),50);
N2=(2/pi)*(asin(T./A2)+(T./A2).*sqrt(1-(T./A2).^2));
figure(5);clf
plot(A2,-1./N2,[0 5]',[real(GG2) real(GG2)],'k--');grid on
axis([1 1.75 -2 -0.9]);
if real(GG2) < -1
    Asat=fsolve('Nsat',[1]);
else
    Asat=inf;
end

figure(2);clf
plot(real(GG),imag(GG));axis([-8 .1 -4 4]);
grid on;hold on;
plot(real(-1./N2),imag(-1./N2),'r');
if real(GG2) < -1
    plot(real(GG2),imag(GG2),'ro');
end
hold off;
xlabel('Real');ylabel('Imag');
h=legend({'$G$','$-\frac{1}{N}$'},'Location','NorthEast','interpreter','latex');
title(['With a Saturation , K=',num2str(K),' A=',num2str(Asat)])

Delta=T/3;
A3=logspace(log10(Delta),1,200);
N3=(4*T./(A3*pi)).*(sqrt(1-(Delta./A3).^2)-sqrt(-1)*Delta./A3);
figure(3);clf
plot(real(GG),imag(GG));
axis([-5 .1 -1 1]);
grid on;hold on;
plot(real(-1./N3),imag(-1./N3),'r');
```

```
ii=find(abs(imag(GG) + (Delta*pi)/(4*T)) < .05)
GG3=GG(ii);Ahyst=fsolve('Nhyst',[2])
plot(real(GG3),imag(GG3),'ro');
hold off;
xlabel('Real');ylabel('Imag');
h=legend({'$G$','$-\frac{1}{N}$'},'Location','NorthEast','interpreter','latex');
title(['With a Relay with Hysteresis, K=',num2str(K),' A=',num2str(Ahyst)])

figure(4);clf
A2=[0 A2];N2=[1 N2];
semilogy(A1,N1,A2,N2,'k--');grid on
xlabel('$A$','Interpreter','latex');ylabel('$N$','Interpreter','latex');
legend('Relay','Saturation','Location','NorthEast');
axis([0 5 1e-1 20]);

sim('RL1');sim('RL2');sim('RL3');

figure(6)
plot(RL(:,1),RL(:,2))
hold on
plot(RL2(:,1),RL2(:,2),'g--')
plot(RL3(:,1),RL3(:,2),'r:')
hold off
legend('Relay','Saturation','Hyst','Location','NorthEast');
grid on
xlabel('$x$','Interpreter','latex');
ylabel('$\dot{x}$','Interpreter','latex');
title(['Sim response with different NLs, K=',num2str(K)])

if K==1.5
    figure(1);export_fig G_examp1 -pdf
    figure(2);export_fig G_examp2 -pdf
    figure(3);export_fig G_examp3 -pdf
    figure(4);export_fig  G_examp4 -pdf
    figure(6);export_fig  G_examp6 -pdf
elseif K==1
    figure(1);export_fig G_examp1a -pdf
    figure(2);export_fig G_examp2a -pdf
    figure(3);export_fig G_examp3a -pdf
    figure(6);export_fig G_examp6a -pdf
else
    figure(1);export_fig G_examp1b -pdf
    figure(2);export_fig  G_examp2b -pdf
    figure(3);export_fig  G_examp3b -pdf
    figure(6);export_fig  G_examp6b -pdf
end
```

```
function y=Nsat(A);
global T GG2

N2=(2/pi)*(asin(T/A)+(T/A)*sqrt(1-(T/A)^2));
y=-1/N2-real(GG2);

end
```

```
function y=Nhyst(A);
global T GG3 Delta

neginvN3=-(A*pi/(4*T))*sqrt(1-(Delta/A)^2)-sqrt(-1)*(Delta*pi)/(4*T)
y=real(neginvN3)-real(GG3)

end
```

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