Topic #14

16.30/31 Feedback Control Systems

State-Space Systems

- Open-loop Estimators
- Closed-loop Estimators

- **Observer Theory (no noise) – Luenberger**
  IEEE TAC Vol 16, No. 6, pp. 596–602, Dec 1971.
- **Estimation Theory (with noise) – Kalman**

- Reading: FPE 7.5

Estimators/Observers

- **Problem:** So far we have assumed that we have full access to the state $\mathbf{x}(t)$ when we designed our controllers.
  - Most often all of this information is not available.

- Usually can only feedback information that is developed from the sensors measurements.
  - Could try "output feedback"

$$\mathbf{u} = K\mathbf{x} \quad \Rightarrow \quad \mathbf{u} = \hat{K}\mathbf{y}$$

  - Same as the proportional feedback we looked at at the beginning of the root locus work.
  - This type of control is very difficult to design in general.

- **Alternative approach:** Develop a replica of the dynamic system that provides an "estimate" of the system states based on the measured output of the system.

- **New plan:**
  1. Develop estimate of $\mathbf{x}(t)$ that will be called $\hat{\mathbf{x}}(t)$.
  2. Then switch from $\mathbf{u}(t) = -K\mathbf{x}(t)$ to $\mathbf{u}(t) = -K\hat{\mathbf{x}}(t)$.

- Two key questions:
  - How do we find $\hat{\mathbf{x}}(t)$?
  - Will this new plan work?

Estimation Schemes

- Assume that the system model is of the form:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\ , \quad \mathbf{x}(0) \text{ unknown} \\ \mathbf{y}(t) &= C\mathbf{x}(t)\end{aligned}$$

where

1. A, B, and C are known.
2. $\mathbf{u}(t)$ is known
3. Measurable outputs are $\mathbf{y}(t)$ from $C \neq I$

- **Goal:** Develop a dynamic system whose state

$$\hat{\mathbf{x}}(t) = \mathbf{x}(t)$$

for all time $t \geq 0$. Two primary approaches:

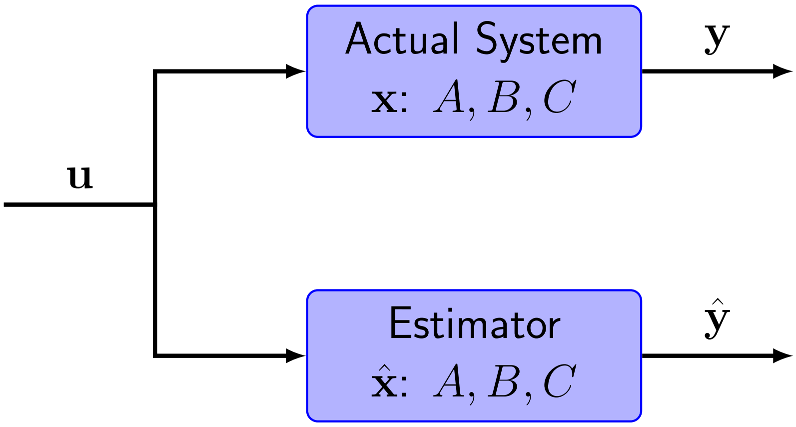
  - Open-loop.
  - Closed-loop.

Open-loop Estimator

- Given that we know the plant matrices and the inputs, we can just perform a simulation that runs in parallel with the system

$$\dot{\hat{\mathbf{x}}}(t) = A\hat{\mathbf{x}}(t) + B\mathbf{u}(t)$$

- Then $\hat{\mathbf{x}}(t) \equiv \mathbf{x}(t)\ \forall\ t$ provided that $\hat{\mathbf{x}}(0) = \mathbf{x}(0)$



- Analysis of this case:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\end{aligned}$$

- Define the **estimation error** $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$.
  Now want $\tilde{\mathbf{x}}(t) = 0\ \forall\ t$. (But is this realistic?)

- **Major Problem:** We do not know $\mathbf{x}(0)$

- Subtract to get:

$$\frac{d}{dt}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) = A(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \quad \Rightarrow \quad \dot{\tilde{\mathbf{x}}}(t) = A\tilde{\mathbf{x}}(t)$$

  which has the solution

$$\tilde{\mathbf{x}}(t) = e^{At}\tilde{\mathbf{x}}(0)$$

  - Gives the estimation error in terms of the initial error.

- Does this guarantee that $\tilde{\mathbf{x}}(t) = 0 \ \forall \ t$?
  Or even that $\tilde{\mathbf{x}}(t) \to 0$ as $t \to \infty$? (which is a more realistic goal).

  - Response is fine if $\tilde{\mathbf{x}}(0) = 0$. But what if $\tilde{\mathbf{x}}(0) \neq 0$?

- If A stable, then $\tilde{\mathbf{x}}(t) \to 0$ as $t \to \infty$, but the dynamics of the estimation error are completely determined by the open-loop dynamics of the system (eigenvalues of A).

  - Could be very slow.

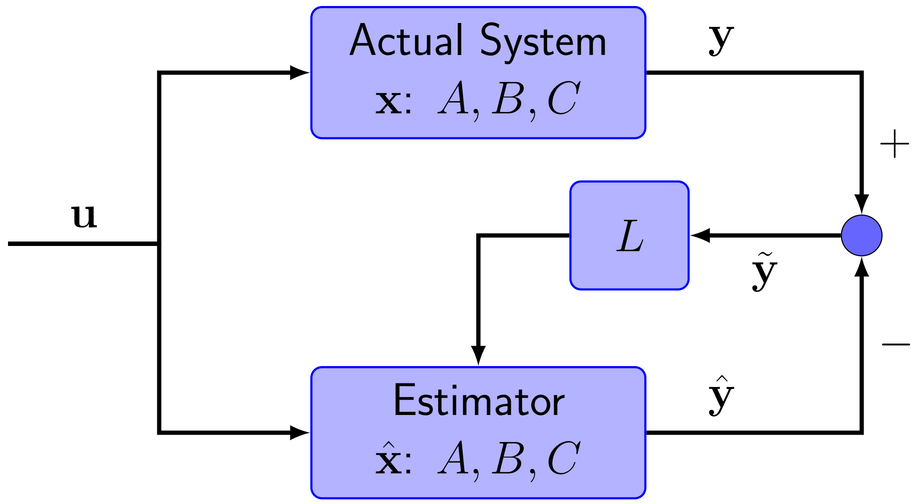
  - No obvious way to modify the estimation error dynamics.

- Open-loop estimation does not seem to be a very good idea.

Closed-loop Estimator

- An obvious way to fix this problem is to use the additional information available:
  - How well does the estimated output match the measured output?

    Compare: $\mathbf{y}(t) = C\mathbf{x}(t)$ with $\hat{\mathbf{y}}(t) = C\hat{\mathbf{x}}(t)$
  - Then form $\tilde{\mathbf{y}}(t) = \mathbf{y}(t) - \hat{\mathbf{y}}(t) \equiv C\tilde{\mathbf{x}}(t)$



- **Approach:** Feedback $\tilde{\mathbf{y}}(t)$ to improve our estimate of the state. Basic form of the estimator is:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + \boxed{L\tilde{\mathbf{y}}(t)} \\ \hat{\mathbf{y}}(t) &= C\hat{\mathbf{x}}(t)\end{aligned}$$

  where L is the *user selectable gain matrix.*

- **Analysis:**

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) \\ &= [A\mathbf{x}(t) + B\mathbf{u}(t)] - [A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + L(\mathbf{y}(t) - \hat{\mathbf{y}}(t))] \\ &= A(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) - L(C\mathbf{x}(t) - C\hat{\mathbf{x}}(t)) \\ &= A\tilde{\mathbf{x}}(t) - LC\tilde{\mathbf{x}}(t) \\ &= (A - LC)\tilde{\mathbf{x}}(t)\end{aligned}$$

- So the closed-loop estimation error dynamics are now

$$\dot{\tilde{\mathbf{x}}}(t) = (A - LC)\tilde{\mathbf{x}}(t)$$

with solution

$$\tilde{\mathbf{x}}(t) = e^{(A-LC)t}\,\tilde{\mathbf{x}}(0)$$

- **Bottom line:** Can select the gain L to attempt to improve the convergence of the estimation error (and/or speed it up).
  - But now must worry about observability of the system model.

- Closed-loop estimator:

$$\begin{aligned}
\dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + L\tilde{\mathbf{y}}(t) \\
&= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + L(\mathbf{y}(t) - \hat{\mathbf{y}}(t)) \\
&= (A - LC)\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + L\mathbf{y}(t) \\
\hat{\mathbf{y}}(t) &= C\hat{\mathbf{x}}(t)
\end{aligned}$$

  - Which is a dynamic system with poles given by $\lambda_i(A - LC)$ and which takes the measured plant outputs as an input and generates an estimate of $\mathbf{x}(t)$.

Regulator/Estimator Comparison

- **Regulator Problem:**
  - Concerned with controllability of (A, B)

    For a controllable system we can place the eigenvalues of $A - BK$ arbitrarily.
  - Choose $K \in \mathbb{R}^{1\times n}$ (SISO) such that the closed-loop poles

    $$\det(sI - A + BK) = \Phi_c(s)$$

    are in the desired locations.

- **Estimator Problem:**
  - For estimation, were concerned with observability of pair (A, C).

    For a observable system we can place the eigenvalues of $A - LC$ arbitrarily.
  - Choose $L \in \mathbb{R}^{n\times 1}$ (SISO) such that the closed-loop poles

    $$\det(sI - A + LC) = \Phi_o(s)$$

    are in the desired locations.

- These problems are obviously very similar – in fact they are called **dual problems**.

Estimation Gain Selection

- The procedure for selecting L is very similar to that used for the regulator design process.

- Write the system model in **observer canonical** form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Now very simple to form

$$\begin{aligned} A - LC &= \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -a_1 - l_1 & 1 & 0 \\ -a_2 - l_2 & 0 & 1 \\ -a_3 - l_3 & 0 & 0 \end{bmatrix} \end{aligned}$$

  - The closed-loop poles of the estimator are at the roots of

    $$\det(sI - A + LC) = s^3 + (a_1 + l_1)s^2 + (a_2 + l_2)s + (a_3 + l_3) = 0$$

  - Use Pole Placement algorithm with this characteristic equation.

- Note: estimator equivalent of Ackermann's formula is that

$$L = \Phi_e(A)\mathcal{M}_o^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Dual Design Approach

- Note that the poles of $(A - LC)$ and $(A - LC)^T$ are identical.
  - Also we have that $(A - LC)^T = A^T - C^T L^T$
  - So designing L^T for this transposed system looks like a standard regulator problem $(A - BK)$ where

$$\begin{aligned} A &\Rightarrow A^T \\ B &\Rightarrow C^T \\ K &\Rightarrow L^T \end{aligned}$$

  So we can use

$$K_e = \texttt{acker}(A^T, C^T, P) \ , \quad L \equiv K_e^T$$

- In fact, just as `k=lqr(A,B,Q,R)` returns a good set of control gains, can use

$$K_e = \texttt{lqr}(A^T, C^T, \tilde{Q}, \tilde{R}) \ , \quad L \equiv K_e^T$$

  to design a good set of "optimal" estimator gains
  - Roles of $\tilde{Q}$ and $\tilde{R}$ explained in 16.322

Estimators Example

- Simple system

$$A = \begin{bmatrix} -1 & 1.5 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 0$$

  - Assume that the initial conditions are not well known.
  - System stable, but $\lambda_{\max}(A) = -0.18$
  - Test observability:

$$\texttt{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \texttt{rank} \begin{bmatrix} 1 & 0 \\ -1 & 1.5 \end{bmatrix}$$

- Use open and closed-loop estimators
  - Since the initial conditions are not well known, use

$$\hat{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Open-loop estimator:

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) \\ \hat{\mathbf{y}}(t) &= C\hat{\mathbf{x}}(t) \end{aligned}$$

- Typically simulate both systems together for simplicity

- Open-loop case:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \mathbf{y}(t) &= C\mathbf{x}(t)\\ \dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t)\\ \hat{\mathbf{y}}(t) &= C\hat{\mathbf{x}}(t)\end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} \mathbf{u}(t)$$

$$\begin{bmatrix} \mathbf{x}(0) \\ \hat{\mathbf{x}}(0) \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{y}(t) \\ \hat{\mathbf{y}}(t) \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix}$$

- Closed-loop case:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t)\\ \dot{\hat{\mathbf{x}}}(t) &= (A - LC)\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + LC\mathbf{x}(t)\end{aligned}$$

$$\Rightarrow \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\hat{\mathbf{x}}}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \hat{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} \mathbf{u}(t)$$

- Example uses a strong $\mathbf{u}(t)$ to shake things up

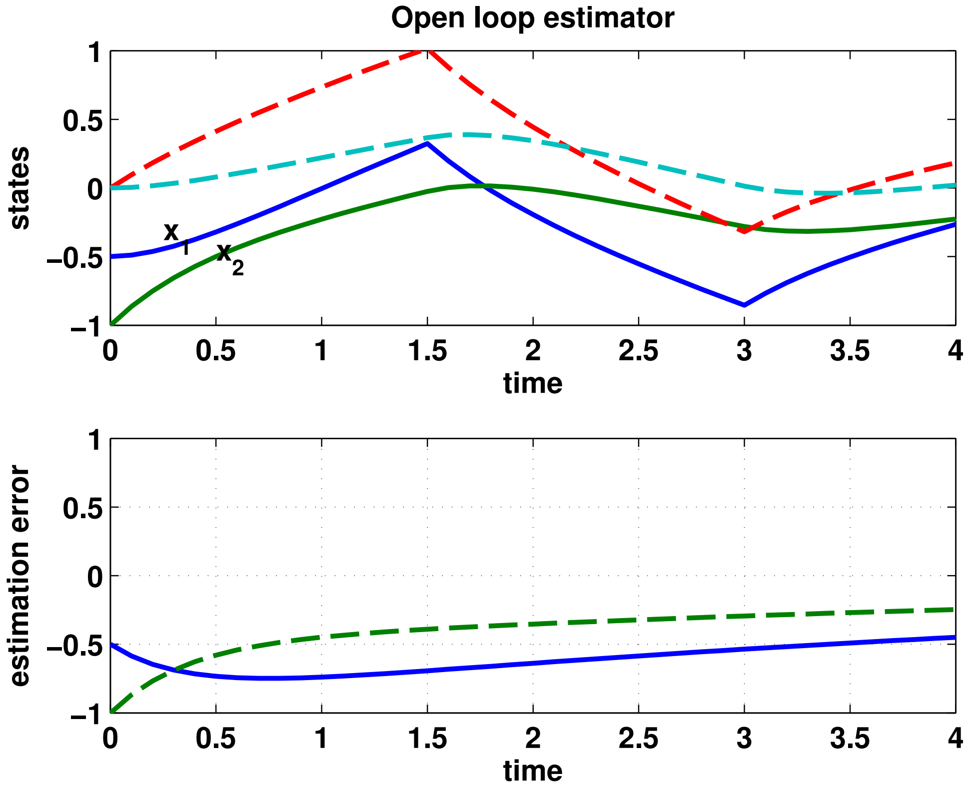


Fig. 1: Open-loop estimator error converges to zero, but very slowly.

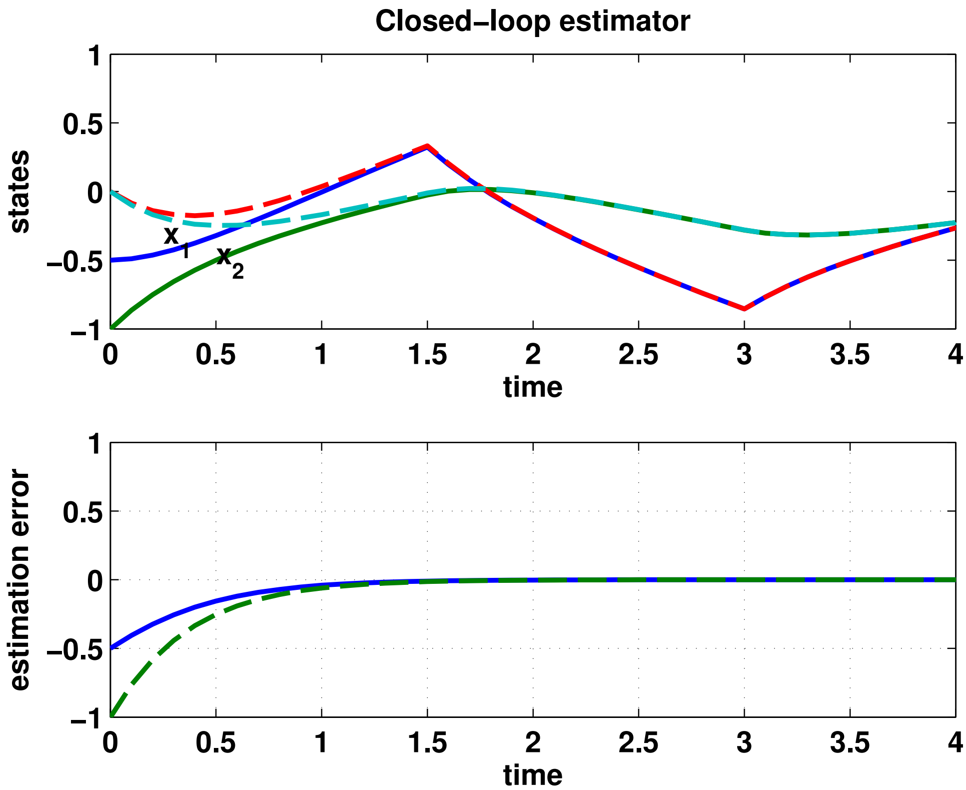


Fig. 2: Closed-loop estimator. Convergence looks much better.

Where Put Estimator Poles?

- Location heuristics for poles still apply
  - Main difference: probably want to make the estimator faster than you intend to make the regulator – should enhance the control, which is based on $\hat{\mathbf{x}}(t)$.
  - Crude ROT: Factor of ≈2 in the time constant $\zeta\omega_n$ associated with the regulator poles.

- **Note:** When designing a regulator, were concerned with "bandwidth" of the control getting too high ⇒ often results in control commands that *saturate* the actuators and/or change rapidly.

- Different concerns for the estimator:
  - Loop closed inside computer, so saturation not a problem.
  - However, measurements $\mathbf{y}$ are often "noisy", and must be careful how we use them to develop state estimates.

⇒ **High bandwidth estimators** tend to accentuate the effect of sensing noise in the estimate.

- State estimates tend to "track" the data in the measurements, which could be fluctuating randomly due to the noise.

⇒ **Low bandwidth estimators** have lower gains and tend to rely more heavily on the plant model

- Essentially an open-loop estimator – tends to ignore the measurements and just uses the plant model.

Final Thoughts

- Note that the feedback gain L in the estimator only stabilizes the estimation error.
  - If the system is unstable, then the state estimates will also go to ∞, with zero error from the actual states.

- Estimation is an important concept of its own.
  - Not always just "part of the control system"
  - Critical issue for guidance and navigation system

- Can develop an optimal estimate as well
  - More complete discussion requires that we study stochastic processes and optimization theory.
  - More in 16.322 – take in Spring or see 2004 OCW notes

- Estimation is all about **which do you trust more**: your measurements or your model.

Code: Estimator

```matlab
% Examples of estimator performance
%
% Jonathan How
% Oct 2010
%
% plant dynamics
%
close all;clear all
set(0,'DefaultLineLineWidth',2);
set(0,'DefaultlineMarkerSize',10);set(0,'DefaultlineMarkerFace','b')
set(0, 'DefaultAxesFontSize', 12);set(0, 'DefaultTextFontSize', 12)

set(0,'DefaultFigureColor','w',...
      'DefaultAxesColor','w',...
      'DefaultAxesXColor','k',...
      'DefaultAxesYColor','k',...
      'DefaultAxesZColor','k',...
      'DefaultTextColor','k')

a=[-1 1.5;1 -2];b=[1 0]';c=[1 0];d=0;
%
% estimator gain calc
l=place(a',c',[-3 -4]);l=l'
%
% plant initial cond
xo=[-.5;-1];
% extimator initial cond
xe=[0 0]';
%
t=[0:.1:10];
%
% inputs
u=0;u=[ones(15,1);-ones(15,1);ones(15,1)/2;-ones(15,1)/2;zeros(41,1)];
%
% open-loop extimator
A_ol=[a zeros(size(a));zeros(size(a)) a];
B_ol=[b;b];
C_ol=[c zeros(size(c));zeros(size(c)) c];
D_ol=zeros(2,1);
%
% closed-loop extimator
A_cl=[a zeros(size(a));l*c a-l*c];
B_cl=[b;b];
C_cl=[c zeros(size(c));zeros(size(c)) c];
D_cl=zeros(2,1);

[y_cl,x_cl]=lsim(A_cl,B_cl,C_cl,D_cl,u,t,[xo;xe]);
[y_ol,x_ol]=lsim(A_ol,B_ol,C_ol,D_ol,u,t,[xo;xe]);

figure(1);clf;subplot(211)
set(gca)
plot(t,x_cl(:,[1 2]),t,x_cl(:,[3 4]),'--','LineWidth',2);axis([0 4 -1 1]);
title('Closed-loop estimator');ylabel('states');xlabel('time')
text(.25,-.4,'x_1');text(.5,-.55,'x_2');subplot(212)
plot(t,x_cl(:,[1 2])-x_cl(:,[3 4]))
setlines(2);axis([0 4 -1 1]);grid on
ylabel('estimation error');xlabel('time')

figure(2);clf;subplot(211)
set(gca)
plot(t,x_ol(:,[1 2]),t,x_ol(:,[3 4]),'--','LineWidth',2);axis([0 4 -1 1])
title('Open loop estimator');ylabel('states');xlabel('time')
text(.25,-.4,'x_1');text(.5,-.55,'x_2');subplot(212)
plot(t,x_ol(:,[1 2])-x_ol(:,[3 4]))
setlines(2);axis([0 4 -1 1]);grid on
ylabel('estimation error');xlabel('time')

figure(1);export_fig est11 -pdf
figure(2);export_fig est12 -pdf
```

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