Topic #10

16.30/31 Feedback Control Systems

State-Space Systems

- **State-space model features**
- Controllability

Controllability

- **Definition:** An LTI **system** is **controllable** if, for every $\mathbf{x}^\star(t)$ and every finite $T > 0$, there exists an input function $\mathbf{u}(t)$, $0 < t \leq T$, such that the system state goes from $\mathbf{x}(0) = 0$ to $\mathbf{x}(T) = \mathbf{x}^\star$.

  - Starting at 0 is not a special case – if we can get to any state in finite time from the origin, then we can get from any initial condition to that state in finite time as well. [1]

- This definition of controllability is consistent with the notion we used before of being able to "influence" all the states in the system in the decoupled examples (page 9–??).

  - ROT: For those decoupled examples, if part of the state cannot be "influenced" by $\mathbf{u}(t)$, then it would be impossible to move that part of the state from 0 to $\mathbf{x}^\star$

- Need only consider the forced solution to study controllability.

$$\mathbf{x}_f(t) = \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau$$

  - Change of variables $\tau_2 = t - \tau$, $d\tau = -d\tau_2$ gives a form that is a little easier to work with:

$$\mathbf{x}_f(t) = \int_0^t e^{A\tau_2} B\mathbf{u}(t - \tau_2) d\tau_2$$

  - Assume system has m inputs.

[1] This controllability from the origin is often called **reachability**.

- Note that, regardless of the eigenstructure of A, the Cayley-Hamilton theorem gives

$$e^{At} = \sum_{i=0}^{n-1} A^i \alpha_i(t)$$

for some computable scalars $\alpha_i(t)$, so that

$$\mathbf{x}_f(t) = \sum_{i=0}^{n-1} (A^i B) \int_0^t \alpha_i(\tau_2) \mathbf{u}(t - \tau_2) d\tau_2 = \sum_{i=0}^{n-1} (A^i B) \boldsymbol{\beta}_i(t)$$

for coefficients $\boldsymbol{\beta}_i(t)$ that depend on the input $\mathbf{u}(\tau)$, $0 < \tau \leq t$.

- Result can be interpreted as meaning that the state $\mathbf{x}_f(t)$ is a linear combination of the nm vectors $A^i B$ (with m inputs).
  - All linear combinations of these nm vectors is the *range space* of the matrix formed from the $A^i B$ column vectors:

$$\mathcal{M}_c = \left[\, B \;\; AB \;\; A^2 B \;\; \cdots \;\; A^{n-1} B \,\right]$$

- **Definition:** Range space of M_c is **controllable subspace** of the system
  - If a state $\mathbf{x}_c(t)$ is not in the range space of M_c, it is not a linear combination of these columns $\Rightarrow$ it is impossible for $\mathbf{x}_f(t)$ to ever equal $\mathbf{x}_c(t)$ – called **uncontrollable state.**

- **Theorem: LTI system is controllable iff it has no uncontrollable states.**
  - Necessary and sufficient condition for controllability is that

$$\texttt{rank}\ \mathcal{M}_c \triangleq \texttt{rank} \left[\, B \;\; AB \;\; A^2 B \;\; \cdots \;\; A^{n-1} B \,\right] = n$$

Further Examples

- With Model # 2:

$$
\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{\mathbf{x}} \\
\mathcal{M}_0 &= \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -6 & 0 \end{bmatrix} \\
\mathcal{M}_c &= \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}
\end{aligned}
$$

  - $\texttt{rank}\ \mathcal{M}_0 = 1$ and $\texttt{rank}\ \mathcal{M}_c = 2$
  - So this model of the system is controllable, but not observable.

- With Model # 3:

$$
\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 3 & 2 \end{bmatrix} \bar{\mathbf{x}} \\
\mathcal{M}_0 &= \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -6 & -2 \end{bmatrix} \\
\mathcal{M}_c &= \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

  - $\texttt{rank}\ \mathcal{M}_0 = 2$ and $\texttt{rank}\ \mathcal{M}_c = 1$
  - So this model of the system is observable, but not controllable.

- Note that controllability/observability are **not** intrinsic properties of a system. Whether the model has them or not depends on the representation that you choose.
  - But they indicate that something else more fundamental is wrong. . .

Modal Tests

- Earlier examples showed the relative simplicity of testing observability/controllability for system with a *decoupled* A matrix.

- There is, of course, a very special decoupled form for the state-space model: the **Modal Form** (6–??)

- Assuming that we are given the model

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u}\end{aligned}$$

and the A is diagonalizable ($A = T\Lambda T^{-1}$) using the transformation

$$T = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

based on the eigenvalues of A. Note that we wrote:

$$T^{-1} = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

which is a column of rows.

- Then define a new state so that $\mathbf{x} = T\mathbf{z}$, then

$$\begin{aligned}\dot{\mathbf{z}} &= T^{-1}\dot{\mathbf{x}} = T^{-1}(A\mathbf{x} + B\mathbf{u}) \\ &= (T^{-1}AT)\mathbf{z} + T^{-1}B\mathbf{u} \\ &= \Lambda\mathbf{z} + T^{-1}B\mathbf{u}\end{aligned}$$

$$\begin{aligned}\mathbf{y} &= C\mathbf{x} + D\mathbf{u} \\ &= CT\mathbf{z} + D\mathbf{u}\end{aligned}$$

- The new model in the state $\mathbf{z}$ is diagonal. There is no coupling in the dynamics matrix Λ.

- But by definition,

$$T^{-1}B = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix} B$$

  and

$$CT = C \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

- Thus if it turned out that

$$w_i^T B \equiv 0$$

  then that element of the state vector z_i would be **uncontrollable** by the input u.

- Also, if

$$Cv_j \equiv 0$$

  then that element of the state vector z_j would be **unobservable** with this sensor.

- Thus, **all modes of the system are controllable and observable** if it can be shown that

$$w_i^T B \neq 0 \ \ \forall\ i$$

  and

$$Cv_j \neq 0 \ \ \forall\ j$$

Cancelation

- Examples show the close connection between pole-zero cancelation and loss of observability and controllability. Can be strengthened.

- **Theorem:** The mode (λ_i, v_i) of a system (A, B, C, D) is unobservable iff the system has a zero at λ_i with direction $\begin{bmatrix} v_i \\ 0 \end{bmatrix}$.

- **Proof:** If the system is unobservable at λ_i, then we know

$$\begin{aligned} (\lambda_i I - A)v_i &= 0 \quad \text{It is a mode} \\ C v_i &= 0 \quad \text{That mode is unobservable} \end{aligned}$$

  Combine to get:

$$\begin{bmatrix} (\lambda_i I - A) \\ C \end{bmatrix} v_i = 0$$

  Or

$$\begin{bmatrix} (\lambda_i I - A) & -B \\ C & D \end{bmatrix} \begin{bmatrix} v_i \\ 0 \end{bmatrix} = 0$$

  which implies that the system has a zero at that frequency as well, with direction $\begin{bmatrix} v_i \\ 0 \end{bmatrix}$.

- Can repeat the process looking for loss of controllability, but now using zeros with left direction $\begin{bmatrix} w_i^T & 0 \end{bmatrix}$.

- **Combined Definition:** when a MIMO zero causes loss of either observability or controllability we say that there is a pole/zero cancelation.

  - MIMO pole-zero (right direction generalized eigenvector) cancelation $\Leftrightarrow$ mode is unobservable

  - MIMO pole-zero (left direction generalized eigenvector) cancelation $\Leftrightarrow$ mode is uncontrollable

- **Note:** This cancelation requires an agreement of both the frequency and the directionality of the system mode (eigenvector) and zero $\begin{bmatrix} v_i \\ 0 \end{bmatrix}$ or $\begin{bmatrix} w_i^T & 0 \end{bmatrix}$.

Connection to Residue

- Recall that in modal form, the state-space model (assumes diagonalizable) is given by the matrices

$$A = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix} \quad B = \begin{bmatrix} w_1^T B \\ \vdots \\ w_n^T B \end{bmatrix} \quad C = \begin{bmatrix} Cv_1 & \cdots & Cv_n \end{bmatrix}$$

for which case it can easily be shown that

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} Cv_1 & \cdots & Cv_n \end{bmatrix} \begin{bmatrix} \frac{1}{s-p_1} & & \\ & \ddots & \\ & & \frac{1}{s-p_n} \end{bmatrix} \begin{bmatrix} w_1^T B \\ \vdots \\ w_n^T B \end{bmatrix} \\ &= \sum_{i=1}^{n} \frac{(Cv_i)(w_i^T B)}{s - p_i} \end{aligned}$$

- Thus the **residue** of each pole is a direct function of the product of the *degree* of controllability and observability for that mode.
  - Loss of observability or controllability ⇒ residue is zero ⇒ that pole does not show up in the transfer function.
  - If modes have equal observability $Cv_i \approx Cv_j$, but one

    $$w_i^T B \gg w_j^T B$$

    then the residue of the i$^{\text{th}}$ mode will be much larger.
  - Great way to approach model reduction if needed.

Weaker Conditions

- Often it is too much to assume that we will have full observability and controllability. Often have to make do with the following. System called:
  - **Detectable** if all unstable modes are **observable**
  - **Stabilizable** if all unstable modes are **controllable**

- So if you had a stabilizable and detectable system, there could be dynamics that you are not aware of and cannot influence, but you know that they are at least stable.

- That is enough information on the system model for now – will assume minimal models from here on and start looking at the control issues.

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