Topic #9

16.30/31 Feedback Control Systems

State-Space Systems

- **State-space model features**
- Observability
- Controllability
- Minimal Realizations

State-Space Model Features

- There are some key characteristics of a state-space model that we need to identify.
  - Will see that these are very closely associated with the concepts of pole/zero cancelation in transfer functions.

- **Example:** Consider a simple system

$$G(s) = \frac{6}{s+2}$$

for which we develop the state-space model

$$\text{Model \# 1} \qquad \begin{aligned} \dot{x} &= -2x + 2u \\ y &= 3x \end{aligned}$$

- But now consider the new state space model $\bar{\mathbf{x}} = [x \;\; x_2]^T$

$$\text{Model \# 2} \qquad \begin{aligned} \dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{\mathbf{x}} \end{aligned}$$

which is clearly different than the first model, and larger.

- But let's looks at the transfer function of the new model:

$$\begin{aligned} \bar{G}(s) &= C(sI - A)^{-1}B + D \\ &= \begin{bmatrix} 3 & 0 \end{bmatrix} \left(sI - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{s+2} \\ \frac{1}{s+1} \end{bmatrix} = \frac{6}{s+2} \quad !! \end{aligned}$$

- This is a bit strange, because previously our figure of merit when comparing one state-space model to another (page 6–??) was whether they reproduced the same same transfer function
  - Now we have two very different models that result in the same transfer function
  - Note that I showed the second model as having 1 extra state, but I could easily have done it with 99 extra states!!

- So what is going on?
  - A clue is that the dynamics associated with the second state of the model x_2 were eliminated when we formed the product

$$\bar{G}(s) = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{s+2} \\ \frac{1}{s+1} \end{bmatrix}$$

  because the A is decoupled and there is a zero in the C matrix

  - Which is exactly the same as saying that there is a **pole-zero cancelation** in the transfer function $\tilde{G}(s)$

$$\frac{6}{s+2} = \frac{6(s+1)}{(s+2)(s+1)} \triangleq \tilde{G}(s)$$

  - Note that model #2 is one possible state-space model of $\tilde{G}(s)$ (has 2 poles)

- For this system we say that the dynamics associated with the second state are **unobservable** using this sensor (defines C matrix).
  - There could be a lot "motion" associated with x_2, but we would be unaware of it using this sensor.

- There is an analogous problem on the input side as well. Consider:

$$\text{Model \# 1} \qquad \begin{aligned} \dot{x} &= -2x + 2u \\ y &= 3x \end{aligned}$$

with $\bar{\mathbf{x}} = [\, x \;\; x_2]^T$

$$\text{Model \# 3} \qquad \begin{aligned} \dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 2 \end{bmatrix} \bar{\mathbf{x}} \end{aligned}$$

which is also **clearly different** than model #1, and has a different form from the second model.

$$\begin{aligned} \hat{G}(s) &= \begin{bmatrix} 3 & 2 \end{bmatrix} \left(sI - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{s+2} & \frac{2}{s+1} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{6}{s+2} \quad !! \end{aligned}$$

- Once again the dynamics associated with the pole at $s = -1$ are canceled out of the transfer function.
  - But in this case it occurred because there is a 0 in the B matrix

- So in this case we can "see" the state x_2 in the output $C = \begin{bmatrix} 3 & 2 \end{bmatrix}$, but we cannot "influence" that state with the input since

$$B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

- So we say that the dynamics associated with the second state are **uncontrollable** using this actuator (defines the B matrix).

- Of course it can get even worse because we could have

$$\begin{aligned}\dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{\mathbf{x}}\end{aligned}$$

- So now we have

$$\begin{aligned}\widetilde{G(s)} &= \begin{bmatrix} 3 & 0 \end{bmatrix} \left(sI - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{s+2} & \frac{0}{s+1} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \frac{6}{s+2} \quad !!\end{aligned}$$

- Get same result for the transfer function, but now the dynamics associated with x_2 are both unobservable and uncontrollable.

- **Summary:** Dynamics in the state-space model that are **uncontrollable**, **unobservable**, or **both** do not show up in the transfer function.

- Would like to develop models that **only have** dynamics that are both **controllable** and **observable**
  $\Rrightarrow$ called a **minimal realization**
  - A state space model that has the lowest possible order for the given transfer function.

- But first need to develop tests to determine if the models are observable and/or controllable

Observability

- **Definition:** An LTI **system** is **observable** if the initial state $\mathbf{x}(0)$ can be **uniquely deduced** from the knowledge of the input $\mathbf{u}(t)$ and output $\mathbf{y}(t)$ for all t between 0 and any finite $T > 0$.
  - If $\mathbf{x}(0)$ can be deduced, then we can reconstruct $\mathbf{x}(t)$ exactly because we know $\mathbf{u}(t) \Rrightarrow$ we can find $\mathbf{x}(t)\ \forall\ t$.
  - Thus we need only consider the zero-input (homogeneous) solution to study observability.

$$\mathbf{y}(t) = Ce^{At}\mathbf{x}(0)$$

- This definition of observability is consistent with the notion we used before of being able to "see" all the states in the output of the decoupled examples
  - ROT: For those decoupled examples, if part of the state cannot be "seen" in $\mathbf{y}(t)$, then it would be impossible to deduce that part of $\mathbf{x}(0)$ from the outputs $\mathbf{y}(t)$.

- **Definition:** A **state** $\mathbf{x}^\star \neq 0$ is said to be **unobservable** if the zero-input solution $\mathbf{y}(t)$, with $\mathbf{x}(0) = \mathbf{x}^\star$, is zero for all $t \geq 0$
  - Equivalent to saying that $\mathbf{x}^\star$ is an unobservable state if

$$Ce^{At}\mathbf{x}^\star = 0 \;\forall\; t \;\geq 0$$

- For the problem we were just looking at, consider Model #2 with $\mathbf{x}^\star = [\; 0 \;\; 1 \;]^T \neq 0$, then

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 3 & 0 \end{bmatrix} \bar{\mathbf{x}} \end{aligned}$$

  so

$$\begin{aligned} Ce^{At}\mathbf{x}^\star &= \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \;\forall\; t \end{aligned}$$

  So, $\mathbf{x}^\star = [\; 0 \;\; 1 \;]^T$ is an unobservable state for this system.

- But that is as expected, because we knew there was a problem with the state x_2 from the previous analysis

- **Theorem: An LTI system is observable iff it has no unobservable states.**
  - We normally just say that the **pair (A,C) is observable.**

- **Pseudo-Proof:** Let $\mathbf{x}^\star \neq 0$ be an unobservable state and compute the outputs from the initial conditions $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0) = \mathbf{x}_1(0) + \mathbf{x}^\star$
  - Then
  $$\mathbf{y}_1(t) = Ce^{At}\mathbf{x}_1(0) \quad \text{and} \quad \mathbf{y}_2(t) = Ce^{At}\mathbf{x}_2(0)$$
  but
  $$\begin{aligned} \mathbf{y}_2(t) &= Ce^{At}(\mathbf{x}_1(0) + \mathbf{x}^\star) = Ce^{At}\mathbf{x}_1(0) + Ce^{At}\mathbf{x}^\star \\ &= Ce^{At}\mathbf{x}_1(0) = \mathbf{y}_1(t) \end{aligned}$$
  - Thus 2 different initial conditions give the same output $\mathbf{y}(t)$, so it would be impossible for us to deduce the actual initial condition of the system $\mathbf{x}_1(t)$ or $\mathbf{x}_2(t)$ given $\mathbf{y}_1(t)$

- Testing system observability by searching for a vector $\mathbf{x}(0)$ such that $Ce^{At}\mathbf{x}(0) = 0 \ \forall\ t$ is feasible, but very hard in general.
  - Better tests are available.

- **Theorem: The vector $\mathbf{x}^\star$ is an unobservable state iff**

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \mathbf{x}^\star = 0$$

- **Pseudo-Proof:** If $\mathbf{x}^\star$ is an unobservable state, then by definition,

$$Ce^{At}\mathbf{x}^\star = 0 \quad \forall\ t \geq 0$$

But all the derivatives of Ce^{At} exist and for this condition to hold, all derivatives must be zero at $t = 0$. Then

$$Ce^{At}\mathbf{x}^\star\big|_{t=0} = 0 \ \Rightarrow\ C\mathbf{x}^\star = 0$$

$$\left.\frac{d}{dt}Ce^{At}\mathbf{x}^\star\right|_{t=0} = 0 \ \Rightarrow\ CAe^{At}\mathbf{x}^\star\big|_{t=0} = CA\mathbf{x}^\star = 0$$

$$\left.\frac{d^2}{dt^2}Ce^{At}\mathbf{x}^\star\right|_{t=0} = 0 \ \Rightarrow\ CA^2e^{At}\mathbf{x}^\star\big|_{t=0} = CA^2\mathbf{x}^\star = 0$$

$$\vdots$$

$$\left.\frac{d^k}{dt^k}Ce^{At}\mathbf{x}^\star\right|_{t=0} = 0 \ \Rightarrow\ CA^ke^{At}\mathbf{x}^\star\big|_{t=0} = CA^k\mathbf{x}^\star = 0$$

- We only need retain up to the $n-1^{\text{th}}$ derivative because of the *Cayley-Hamilton* theorem.

- **Simple test:** Necessary and sufficient condition for observability is that

$$\texttt{rank}\ \mathcal{M}_o \triangleq \texttt{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

- Why does this make sense?
  - The requirement for an unobservable state is that for $\mathbf{x}^\star \neq 0$
  $$\mathcal{M}_o \mathbf{x}^\star = 0$$
  - Which is equivalent to saying that $\mathbf{x}^\star$ is orthogonal to each row of $\mathcal{M}_o$.
  - But if the rows of $\mathcal{M}_o$ are considered to be vectors and these **span the full n-dimensional space**, then it is not possible to find an n-vector $\mathbf{x}^\star$ that is orthogonal to each of these.
  - To determine if the n rows of $\mathcal{M}_o$ span the full n-dimensional space, we need to test their **linear independence**, which is equivalent to the `rank` test[1]

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[1] Let M be a $m \times p$ matrix, then the `rank` of M satisfies:

1. `rank` $M \equiv$ number of linearly independent columns of M
2. `rank` $M \equiv$ number of linearly independent rows of M
3. `rank` $M \leq \min\{m, p\}$