Topic #8

16.30/31 Feedback Control Systems

State-Space Systems

- System Zeros
- Transfer Function Matrices for MIMO systems

Zeros in State Space Models

- Roots of transfer function numerator called the **system zeros**.
  - Need to develop a similar way of defining/computing them using a state space model.

- **Zero:** generalized frequency s_0 for which the system can have a non-zero input $\mathbf{u}(t) = \mathbf{u}_0 e^{s_0 t}$, but exactly zero output $\mathbf{y}(t) \equiv 0 \ \forall t$
  - Note that there is a specific initial condition associated with this response $\mathbf{x}_0$, so the state response is of the form $\mathbf{x}(t) = \mathbf{x}_0 e^{s_0 t}$

$$\mathbf{u}(t) = \mathbf{u}_0 e^{s_0 t} \quad \Rightarrow \quad \mathbf{x}(t) = \mathbf{x}_0 e^{s_0 t} \quad \Rightarrow \mathbf{y}(t) \equiv 0$$

- Given $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, substitute the above to get:

$$\mathbf{x}_0 s_0 e^{s_0 t} = A\mathbf{x}_0 e^{s_0 t} + B\mathbf{u}_0 e^{s_0 t} \quad \Rightarrow \quad \begin{bmatrix} s_0 I - A & -B \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} = 0$$

- Also have that $\mathbf{y} = C\mathbf{x} + D\mathbf{u} = 0$ which gives:

$$C\mathbf{x}_0 e^{s_0 t} + D\mathbf{u}_0 e^{s_0 t} = 0 \quad \rightarrow \quad \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} = 0$$

- So we must find the s_0 that solves:

$$\begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix} = 0$$

  - Is a **generalized eigenvalue problem** that can be solved in MATLAB using `eig.m` or `tzero.m` [1]

[1] MATLAB is a trademark of the Mathworks Inc.

- There is a zero at the frequency s_0 if there exists a non-trivial solution of

$$\det \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} = 0$$

  - Compare with equation on page 6–??

- **Key Point:** Zeros have both direction $\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \end{bmatrix}$ and frequency s_0
  - Just as we would associate a direction (eigenvector) with each pole (frequency λ_i)

- Example: $G(s) = \frac{s+2}{s^2+7s+12}$

$$A = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad D = 0$$

$$\begin{aligned} \det \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} &= \det \begin{bmatrix} s_0 + 7 & 12 & -1 \\ -1 & s_0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \\ &= (s_0+7)(0) + 1(2) + 1(s_0) = s_0 + 2 = 0 \end{aligned}$$

so there is clearly a zero at $s_0 = -2$, as we expected. For the directions, solve:

$$\begin{bmatrix} s_0 + 7 & 12 & -1 \\ -1 & s_0 & 0 \\ 1 & 2 & 0 \end{bmatrix}_{s_0=-2} \begin{bmatrix} x_{01} \\ x_{02} \\ u_0 \end{bmatrix} = \begin{bmatrix} 5 & 12 & -1 \\ -1 & -2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \\ u_0 \end{bmatrix} = 0?$$

gives $x_{01} = -2x_{02}$ and $u_0 = 2x_{02}$ so that with $x_{02} = 1$

$$\mathbf{x}_0 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{and} \quad u = 2e^{-2t}$$

- Further observations: apply the specified control input in the frequency domain, so that

$$Y_1(s) = G(s)U(s)$$

where $u = 2e^{-2t}$, so that $U(s) = \frac{2}{s+2}$

$$Y_1(s) = \frac{s+2}{s^2+7s+12} \cdot \frac{2}{s+2} = \frac{2}{s^2+7s+12}$$

Say that $s = -2$ is a **blocking zero** or a **transmission zero.**

- The response $Y_1(s)$ is clearly non-zero, but it does not contain a component at the input frequency $s = -2$.
  - That input has been "blocked".

- Note that the output response left in $Y_1(s)$ is of a very special form – it corresponds to the (negative of the) response you would see from the system with $u(t) = 0$ and $\mathbf{x}_0 = \begin{bmatrix} -2 & 1 \end{bmatrix}^T$

$$\begin{aligned} Y_2(s) &= C(sI - A)^{-1}\mathbf{x}_0 \\ &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} s+7 & 12 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} s & -12 \\ 1 & s+7 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{1}{s^2+7s+12} \\ &= \frac{-2}{s^2+7s+12} \end{aligned}$$

- So then the total output is $Y(s) = Y_1(s) + Y_2(s)$ showing that $Y(s) = 0 \rightarrow y(t) = 0$, as expected.

Simpler Test

- Simpler test using **transfer function matrix:**
  - If z is a zero with (right) direction $[\zeta^T\ ,\ \tilde{u}^T\ ]^T$, then
  $$\begin{bmatrix} zI - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \zeta \\ \tilde{u} \end{bmatrix} = 0$$
  - If z not an eigenvalue of A, then $\zeta = (zI - A)^{-1}B\tilde{u}$, which gives
  $$\left[C(zI - A)^{-1}B + D\right] \tilde{u} = G(z)\tilde{u} = 0$$
  - Which implies that $G(s)$ loses rank at $s = z$
  - If $G(s)$ is square, can find the zero frequencies by solving:
  $$\det\ \mathbf{G(s)} = \mathbf{0}$$
  - If any of the resulting roots are also eigenvalues of A, need to re-check the generalized eigenvalue matrix condition.

- Need to be very careful when we find MIMO zeros that have the same frequency as the poles of the system, because it is not obvious that a pole/zero cancelation will occur (for MIMO systems).
  - The zeros have a directionality associated with them, and that must "agree" as well, or else you do not get cancelation
  - More on this topic later when we talk about **controllability** and **observability**

Transfer Function Matrix

- Note that the *transfer function matrix* (TFM) notion is a MIMO generalization of the SISO transfer function
  - It is a matrix of transfer functions

$$G(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1m}(s) \\ & \ddots & \\ g_{p1}(s) & \cdots & g_{pm}(s) \end{bmatrix}$$

  - $g_{ij}(s)$ relates input of actuator j to output of sensor i.
  - It is relatively easy to go from a state-space model to a TFM, but not obvious how to go back the other way.

- Simplest approach is to develop a state space model for each element of $g_{ij}(s)$ in the form $A_{ij}, B_{ij}, C_{ij}, D_{ij}$, and then assemble (if TFM is $p \times m$)

$$A = \begin{bmatrix} A_{11} & & & & & \\ & \ddots & & & & \\ & & A_{1m} & & & \\ & & & A_{21} & & \\ & & & & \vdots & \\ & & & & & A_{pm} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & & \\ & \ddots & \\ & & B_{1m} \\ B_{21} & & \\ & \vdots & \\ & & B_{pm} \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & \cdots & C_{1m} & & & & & \\ & & & C_{21} & \ddots & C_{2m} & & \\ & & & & & \vdots & & \\ & & & & & & C_{p1} \cdots & C_{pm} \end{bmatrix} \quad D = [D_{ij}]$$

- One issue is how many poles are needed - this realization might be inefficient (larger than necessary).
  - Related to **McMillan degree**, which for a proper system is the degree of the characteristic polynomial obtained as the least common denominator of all minors of $G(s)$.[2]
  - **Subtle point:** consider a $m \times m$ matrix A, then the standard minors formed by deleting 1 row and column and taking the determinant of the resulting matrix are called the $m-1^{th}$ **order minors** of A.
  - To consider **all** minors of A, must consider all possible orders, i.e. by selecting $j \leq m$ subsets of the rows and columns and taking the resulting determinant.
- Given an $n \times m$ matrix A with entries a_{ij}, a minor of A is the determinant of a smaller matrix formed from its entries by selecting only some of the rows and columns.
  - Let $K = \{\ k_1 \ \ k_2 \ \ \ldots \ \ k_p\ \}$ and $L = \{\ l_1 \ \ l_2 \ \ \ldots \ \ l_p\ \}$ be subsets of $\{1, 2, \ldots, n\}$ and $\{1, 2, \ldots, m\}$, respectively.
  - Indices are chosen so $k_1 < k_2 \cdots < k_p$ and $l_1 < l_2 \cdots < l_p$.
  - **pth order minor** defined by K and L is the determinant [3]

$$[A]_{K,L} = \begin{vmatrix} a_{k_1 l_1} & a_{k_1 l_2} & \ldots a_{k_1 l_p} \\ a_{k_2 l_1} & a_{k_2 l_2} & \ldots a_{k_2 l_p} \\ \vdots & \ddots & \\ a_{k_p l_1} & a_{k_p l_2} & \ldots a_{k_p l_p} \end{vmatrix}$$

  - If $p = m = n$ then the minor is simply the determinant of the matrix.
- In a nutshell what this means is that a 2×2 matrix has 4 order-1 minors **and** 1 order-2 minor to consider.

[2] Lowest order polynomial that can be divided cleanly by all denominators of the minors of $G(s)$.

[3] See here for details

Gilbert's Realization

- One approach: rewrite the TFM as

$$G(s) = \frac{H(s)}{d(s)}$$

  where $d(s)$ is the least common multiple of the denominators of the entries of $G(s)$.

  - Note difference from the discussion about the McMillan degree.
  - $d(s)$ looks like a characteristic equation for this system, but **it is not** $\Rightarrow$ it does not accurately reflect number of poles needed.

- For proper systems for which $d(s)$ has distinct roots, can use Gilbert's realization.

  - Apply a partial fraction expansion to each of the elements of TFM $G(s)$ and collect residues for each distinct pole[4].

$$G(s) = \sum_{i}^{N_m} \frac{R_i}{s - p_i} \quad \text{where} \quad R_i = \lim_{s \to p_i} (s - p_i) G(s)$$

  - Then sum of the ranks of matrices R_i gives the McMillan degree

[4] Generalizations of this Gilbert's realization approach exist if the g_{ij} have repeated roots.

- Can develop a state space realization by analyzing each element of the partial fraction expansion
  - Set $R_i = C_i B_i$, and find appropriate B_i and C_i
  - Form A_i by placing the poles on the diagonal as many times as needed (determined by rank of R_i)
  - Form state space model:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_{N_m} \end{bmatrix} \mathbf{x} + \begin{bmatrix} B_1 \\ \vdots \\ B_{N_m} \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} C_1 & \cdots & C_{N_m} \end{bmatrix} \mathbf{x} \end{aligned}$$

Zero Example 1

- TFM $G(s) = \begin{bmatrix} \dfrac{1}{s+2} & \dfrac{1}{s+2} \\ \\ \dfrac{1}{s-2} & \dfrac{s-2}{(s+1)(s+2)} \end{bmatrix}$

- To compute the McMillan degree for this system, form all minors (4 order-1 and 1 order-2):

$$\left\{ \frac{1}{s+2}, \quad \frac{1}{s+2}, \quad \frac{1}{s-2}, \quad \frac{s-2}{(s+1)(s+2)}, \quad \frac{2-7s}{(s-2)(s+1)(s+2)^2} \right\}$$

- To find LCD (least common multiple of denominators), pull out smallest polynomial that leaves all terms with no denominator:

$$\frac{1}{(s-2)(s+1)(s+2)^2} \{(s-2)(s+1)(s+2), (s-2)(s+1)(s+2),$$
$$(s+1)(s+2^2), \quad (s-2)^2(s+2), \quad 2-7s\}$$

  - So we expect a fourth order system with poles at $s=2$, $s=-2$ (two), and $s=-1$

- Compare with the Gilbert realization, find $d(s)$:

$$G(s) = \frac{1}{(s+1)(s+2)(s-2)} \begin{bmatrix} (s+1)(s-2) & (s+1)(s-2) \\ \\ (s+1)(s+2) & (s-2)^2 \end{bmatrix}$$
$$= \frac{1}{s+1}\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} + \frac{1}{s-2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \frac{1}{(s+2)}\begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$$

  - Note that the rank of the last 2×2 matrix is 2
  - So the system order is 4 - we need to have two poles $s=-2$.

- So the system model for the example is

$$A_1 = [-1] \qquad B_1 = \begin{bmatrix} 0 & -3 \end{bmatrix} \quad C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} \quad C_2 = I_2$$

$$A_3 = [2] \qquad B_3 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad C_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Note, realization model on 8–5 would be 5th order, not 4th.

Code: MIMO Models

```
%
% basic MIMO TFM to SS
%
G=tf({1 1;1 [1 -2]},{[1 2] [1 2];[1 -2] [1 3 2]});

% find residue matrices of the 3 poles
R1=tf([1 1],1)*G;R1=minreal(R1);R1=evalfr(R1,-1)
R2=tf([1 2],1)*G;R2=minreal(R2);R2=evalfr(R2,-2)
R3=tf([1 -2],1)*G;R3=minreal(R3);R3=evalfr(R3,2)

% form SS model for 3 poles using the residue matrices
A1=[-1];B1=R1(2,:);C1=[0 1]';
A2=[-2 0;0 -2];B2=R2;C2=eye(2);
A3=[2];B3=R3(2,:);C3=[0 1]';

% combine submodels
A=zeros(4);A(1:1,1:1)=A1;A(2:3,2:3)=A2;A(4,4)=A3;
B=[B1;B2;B3];
C=[C1 C2 C3];

syms s
Gn=simple(C*inv(s*eye(4)-A)*B);

% alternative is to make a SS model of each g_{ij}
A11=-2;B11=1;C11=1;
A12=-2;B12=1;C12=1;
A21=2;B21=1;C21=1;
A22=[-3 -2;1 0];B22=[2 0]';C22=[0.5 -1];

% and then combine
AA=zeros(5);AA(1,1)=A11;AA(2,2)=A12;AA(3,3)=A21;AA(4:5,4:5)=A22;
BB=[B11 B11*0;B12*0 B12;B21 B21*0;B22*0 B22];
CC=[C11 C12 zeros(1,3);zeros(1,2) C21 C22];
GGn=simple(CC*inv(s*eye(5)-AA)*BB);

Gn,GGn
```

Zero Example 2

- TFM $G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{(s+1)^2} \\ \dfrac{1}{(s+1)^3} & \dfrac{1}{(s+1)^4} \end{bmatrix}$

- McMillan Degree: find all minors of $G(s)$

$$\frac{1}{s+1}, \quad \frac{1}{(s+1)^2}, \quad \frac{1}{(s+1)^3}, \quad \frac{1}{(s+1)^4}, \quad 0$$

- To find LCD (least common multiple of denominators), pull out smallest polynomial that leaves all terms with no denominator:

$$\frac{1}{(s+1)^4}\left\{(s+1)^3, \quad (s+1)^2, \quad (s+1), \quad 1\right\}$$

- So the LCD is $(s+1)^4$ and the McMillan degree is 4 – we expect the minimal state space model to have 4 poles at $s=-1$.

- Gilbert approach as given cannot be applied directly since $d(s) = \frac{1}{(s+1)^4}$ has repeated roots
  - See Matlab code for model development

$$A = \begin{bmatrix} -1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & -3.00 & -1.50 & -0.50 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 2.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & -2.00 & -1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & -4.00 & -1.50 & -1.00 & -0.50 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 4.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.50 & 0.00 \end{bmatrix} \quad B = \begin{bmatrix} 1.00 & 0.00 \\ 0.50 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 1.00 \\ 0.00 & 0.00 \\ 0.00 & 0.50 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \quad D = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}$$

- Note that $\lambda_i(A) = -1$ – there are 10 poles there. So this is clearly not minimal since the order is 10, not the 4 we expected.

- Matlab command `minreal` can be used to convert to a minimal realization.

$$A = \begin{bmatrix} -0.40 & -0.16 & -1.00 & 0.01 \\ 0.32 & -1.49 & -0.06 & 1.07 \\ 0.50 & -1.06 & -1.17 & -0.39 \\ -0.07 & 0.16 & 0.02 & -0.94 \end{bmatrix} \quad B = \begin{bmatrix} 0.23 & -0.02 \\ -0.97 & 0.36 \\ -0.05 & -0.31 \\ 0.01 & -0.75 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.18 & -1.01 & 0.35 & -0.63 \\ -1.11 & -0.29 & 0.43 & -0.28 \end{bmatrix} \quad D = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}$$

  - New model has 6 states removed - so the minimal degree is 4 as expected.

Code: Zeros (zero example1.m)

```
G1=ss(tf({1 1;1 1},{[1 1] conv([1 1],[1 1]);conv([1 1],conv([1 1],[1 1])) ...
    conv([1 1],conv([1 1],conv([1 1],[1 1])))})); %
[a,b,c,d]=ssdata(G1);
latex(a,'%.2f','nomath') %
latex(b,'%.2f','nomath') %
latex(c,'%.2f','nomath') %
latex(d,'%.2f','nomath') %
G2=minreal(G1);[a2,b2,c2,d2]=ssdata(G2);
latex(a2,'%.2f','nomath') %
latex(b2,'%.2f','nomath') %
latex(c2,'%.2f','nomath') %
latex(d2,'%.2f','nomath') %
```

Zero Example 3

- TFM $G(s) = \begin{bmatrix} \dfrac{2s+3}{s^2+3s+2} & \dfrac{3s+5}{s^2+3s+2} \\ \\ \dfrac{-1}{(s+1)} & 0 \end{bmatrix}$
- McMillan Degree: find all minors of $G(s)$

$$\frac{2s+3}{s^2+3s+2}, \quad \frac{3s+5}{s^2+3s+2}, \quad \frac{-1}{(s+1)}, \quad \frac{-(3s+5)}{(s+1)(s^2+3s+2)}$$

- To find LCD, pull out smallest polynomial that leaves all terms with no denominator:

$$\frac{1}{(s^2+3s+2)(s+1)}\{(2s+3)(s+1), \quad (3s+5)(s+1), \quad -(s^2+3s+2), \quad -(3s+5)\}$$

  - So the LCD is $(s^2+3s+2)(s+1) = (s+1)^2(s+2)$
  - The McMillan degree is 3 – we expect the minimal state space model to have 3 poles.

- For Gilbert approach, we rewrite

$$G(s) = \frac{\begin{bmatrix} 2s+3 & 3s+5 \\ -(s+2) & 0 \end{bmatrix}}{(s+1)(s+2)} = \frac{R_1}{s+1} + \frac{R_2}{s+2}$$

where

$$R_1 = \lim_{s\to -1}(s+1)G(s) = \lim_{s\to -1}\begin{bmatrix} \frac{2s+3}{s+2} & \frac{3s+5}{s+2} \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$$

$$R_2 = \lim_{s\to -2}(s+2)G(s) = \lim_{s\to -2}\begin{bmatrix} \frac{2s+3}{s+1} & \frac{3s+5}{s+1} \\ -\frac{s+2}{s+1} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

which also indicates that we will have a third order system with 2 poles at $s = -1$ and 1 at $s = -2$.

- For the state space model, note that

$$R_1 = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = C_1 B_1$$

$$R_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = C_2 B_2$$

giving

$$A = \left[\begin{array}{cc|c} -1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -2 \end{array}\right] \quad B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 1 & 1 \end{array}\right]$$

$$C = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 0 & 0 \end{array}\right]$$

- From Matlab you get:

$$A = \begin{bmatrix} -1.00 & 0.00 & 0.00 \\ 0.00 & -2.00 & 0.00 \\ 0.00 & 0.00 & -1.00 \end{bmatrix} \quad B = \begin{bmatrix} 0.56 & 1.12 \\ 0.35 & 0.35 \\ 0.50 & 0.00 \end{bmatrix}$$

$$C = \begin{bmatrix} 1.79 & 2.83 & 0.00 \\ 0.00 & 0.00 & -2.00 \end{bmatrix} \quad D = \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.00 \end{bmatrix}$$

Code: Zeros (zero example2.m)

```
G1=ss(tf({[2 3] [3 5];-1 0},{[1 3 2] [1 3 2];[1 1] 1})); %
G1=canon(G1,'modal')
[a,b,c,d]=ssdata(G1);
latex(a,'%.2f','nomath') %
latex(b,'%.2f','nomath') %
latex(c,'%.2f','nomath') %
latex(d,'%.2f','nomath') %
```

Summary of Zeros and TFMs

- Great feature of solving for zeros using the generalized eigenvalue matrix condition is that it can be used to find **MIMO zeros** of a system with multiple inputs/outputs.

$$\det \begin{bmatrix} s_0 I - A & -B \\ C & D \end{bmatrix} = 0$$

- Note: we have to be careful how to analyze these TFM's.
  - Just looking at individual transfer functions is **not useful.**
  - Need to look at system as a whole – use the **singular values** of $G(s)$

- Will see later the conditions to determine if the order of a state space model is **minimal**.

16.30 / 16.31 Feedback Control Systems
Fall 2010