Topic #7

16.30/31 Feedback Control Systems

State-Space Systems

- **What are the basic properties of a state-space model, and how do we analyze these?**
- Time Domain Interpretations
- System Modes

Time Response

- Can develop a lot of insight into the system response and how it is modeled by computing the time response $\mathbf{x}(t)$
  - Homogeneous part
  - Forced solution

- **Homogeneous Part**

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad \mathbf{x}(0) \text{ known}$$

  - Take Laplace transform

$$X(s) = (sI - A)^{-1}\mathbf{x}(0)$$

  so that

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right]\mathbf{x}(0)$$

  - But can show

$$\begin{aligned} (sI - A)^{-1} &= \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots \\ \text{so } \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] &= I + At + \frac{1}{2!}(At)^2 + \ldots \\ &= e^{At} \end{aligned}$$

$$\Rightarrow \mathbf{x}(t) = e^{At}\mathbf{x}(0)$$

- e^{At} is a special matrix that we will use many times in this course
  - *Transition matrix* or *Matrix Exponential*
  - Calculate in MATLAB using `expm.m` and not `exp.m` [1]
  - Note that $e^{(A+B)t} = e^{At}e^{Bt}$ iff $AB = BA$

[1] MATLAB is a trademark of the Mathworks Inc.

- Example: $\dot{\mathbf{x}} = A\mathbf{x}$, with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \frac{1}{(s+2)(s+1)}$$

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

- We will say more about e^{At} when we have said more about A (eigenvalues and eigenvectors)
- Computation of $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$ straightforward for a 2-state system
  - More complex for a larger system, see this paper

SS: Forced Solution

- **Forced Solution**
  - Consider a **scalar case:**

$$\begin{aligned} \dot{x} &= ax + bu, \quad x(0) \text{ given} \\ \Rightarrow x(t) &= e^{at}x(0) + \int_0^t e^{a(t-\tau)} bu(\tau) d\tau \end{aligned}$$

  where did this come from?

  1. $\dot{x} - ax = bu$
  2. $e^{-at}\left[\dot{x} - ax\right] = \frac{d}{dt}(e^{-at}x(t)) = e^{-at}bu(t)$
  3. $\int_0^t \frac{d}{d\tau} e^{-a\tau}x(\tau)d\tau = e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau$

- **Forced Solution – Matrix case:**

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

where $\mathbf{x}$ is an n-vector and $\mathbf{u}$ is a m-vector

- Just follow the same steps as above to get

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau$$

and if $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$, then

$$\mathbf{y}(t) = Ce^{At}\mathbf{x}(0) + \int_0^t Ce^{A(t-\tau)} B\mathbf{u}(\tau) d\tau + D\mathbf{u}(t)$$

  - $Ce^{At}\mathbf{x}(0)$ is the initial response
  - $Ce^{A(t)}B$ is the impulse response of the system.

- Have seen the key role of e^{At} in the solution for $\mathbf{x}(t)$
  - Determines the system time response
  - But would like to get more insight!
- Consider what happens if the matrix A is diagonalizable, i.e. there exists a T such that

$$T^{-1}AT = \Lambda \text{ which is diagonal } \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

Then

$$e^{At} = Te^{\Lambda t}T^{-1}$$

where

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

- Follows since $e^{At} = I + At + \frac{1}{2!}(At)^2 + \ldots$ and that $A = T\Lambda T^{-1}$, so we can show that

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}(At)^2 + \ldots \\ &= I + T\Lambda T^{-1}t + \frac{1}{2!}(T\Lambda T^{-1}t)^2 + \ldots \\ &= Te^{\Lambda t}T^{-1} \end{aligned}$$

- This is a simpler way to get the matrix exponential, but how find T and λ?
  - Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

- Recall that the eigenvalues of A are the same as the roots of the characteristic equation (page 6–7)
- λ is an **eigenvalue** of A if

$$\det(\lambda I - A) = 0$$

which is true iff there exists a nonzero v (**eigenvector**) for which

$$(\lambda I - A)v = 0 \quad \Rightarrow \quad Av = \lambda v$$

- Repeat the process to find all of the eigenvectors. Assuming that the n eigenvectors are linearly independent

$$\begin{aligned} Av_i &= \lambda_i v_i \quad i = 1, \ldots, n \\ A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} &= \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \\ AT = T\Lambda \quad \Rightarrow & \quad T^{-1}AT = \Lambda \end{aligned}$$

Jordan Form

- One word of caution: Not all matrices are diagonalizable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \det(sI - A) = s^2$$

only one eigenvalue $s = 0$ (repeated twice). The eigenvectors solve

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = 0$$

eigenvectors are of the form $\begin{bmatrix} r_1 \\ 0 \end{bmatrix}$, $r_1 \neq 0 \rightarrow$ would only be one.

- Need **Jordan Form** to handle the case with repeated roots [2]
  - Jordan form of matrix $A \in \mathbb{R}^{n \times n}$ is block diagonal:

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \quad \text{with} \quad A_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & & 0 \\ \vdots & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{bmatrix}$$

  - Observation: any matrix can be transformed into Jordan form with the eigenvalues of A determining the blocks A_j.

- The matrix exponential of a Jordan form matrix is then given by

$$e^{At} = \begin{bmatrix} e^{A_1 t} & 0 & \cdots & 0 \\ 0 & e^{A_2 t} & & 0 \\ & & \ddots & \\ 0 & 0 & & e^{A_k t} \end{bmatrix} \quad \text{with} \quad e^{A_j t} = \begin{bmatrix} 1 & t & t^2/2! & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & \ddots & t \\ 0 & 0 & & \cdots & 1 \end{bmatrix} e^{\lambda_j t}$$

[2] see book by Strang, Matlab command `Jordan`, or here

EV Mechanics

- Consider $A = \begin{bmatrix} -1 & 1 \\ -8 & 5 \end{bmatrix}$

$$\begin{aligned} (sI - A) &= \begin{bmatrix} s+1 & -1 \\ 8 & s-5 \end{bmatrix} \\ \det(sI - A) &= (s+1)(s-5) + 8 = s^2 - 4s + 3 = 0 \end{aligned}$$

so the eigenvalues are $s_1 = 1$ and $s_2 = 3$

- Eigenvectors $(sI - A)v = 0$

$$(s_1 I - A)v_1 = \begin{bmatrix} s+1 & -1 \\ 8 & s-5 \end{bmatrix}_{s=1} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0$$

$$\begin{bmatrix} 2 & -1 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = 0 \qquad 2v_{11} - v_{21} = 0, \Rightarrow v_{21} = 2v_{11}$$

v_{11} is then arbitrary ($\neq 0$), so set $v_{11} = 1$

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(s_2 I - A)v_2 = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = 0 \qquad 4v_{12} - v_{22} = 0, \Rightarrow v_{22} = 4v_{12}$$

$$v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- Confirm that $Av_i = \lambda_i v_i$

Dynamic Interpretation

- Since $A = T\Lambda T^{-1}$, then

$$e^{At} = Te^{\Lambda t}T^{-1} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

where we have written

$$T^{-1} = \begin{bmatrix} - & w_1^T & - \\ & \vdots & \\ - & w_n^T & - \end{bmatrix}$$

which is a column of rows.

- Multiply this expression out and we get that

$$e^{At} = \sum_{i=1}^{n} e^{\lambda_i t} v_i w_i^T$$

- Assume A diagonalizable, then $\dot{\mathbf{x}} = A\mathbf{x}$, $\mathbf{x}(0)$ given, has solution

$$\begin{aligned} \mathbf{x}(t) &= e^{At}\mathbf{x}(0) = Te^{\Lambda t}T^{-1}\mathbf{x}(0) \\ &= \sum_{i=1}^{n} e^{\lambda_i t} v_i \{w_i^T \mathbf{x}(0)\} \\ &= \sum_{i=1}^{n} e^{\lambda_i t} v_i \beta_i \end{aligned}$$

- State solution is **linear combination** of the system **modes** $v_i e^{\lambda_i t}$

$e^{\lambda_i t}$ – Determines **nature** of the time response

v_i – Determines how each state **contributes** to that mode

β_i – Determines extent to which initial condition **excites** the mode

- Note that the v_i give the relative sizing of the response of each part of the state vector to the response.

$$v_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \quad \text{mode 1}$$

$$v_2(t) = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} e^{-3t} \quad \text{mode 2}$$

- Clearly $e^{\lambda_i t}$ gives the time modulation
  - λ_i real – growing/decaying exponential response
  - λ_i complex – growing/decaying exponential damped sinusoidal

- **Bottom line:** The locations of the eigenvalues determine the pole locations for the system, thus:
  - They determine the stability and/or performance & transient behavior of the system.
  - It is their locations that we will want to modify when we start the control work

Diagonalization with Complex Roots

- If A has complex conjugate eigenvalues, the process is similar but a little more complicated.
- Consider a 2x2 case with A having eigenvalues $a \pm b\mathbf{i}$ and associated eigenvectors e_1, e_2, with $e_2 = \bar{e}_1$. Then

$$\begin{aligned} A &= \left[\, e_1 \,\middle|\, e_2 \,\right] \begin{bmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, e_2 \,\right]^{-1} \\ &= \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right] \begin{bmatrix} a+b\mathbf{i} & 0 \\ 0 & a-b\mathbf{i} \end{bmatrix} \left[\, e_1 \,\middle|\, \bar{e}_1 \,\right]^{-1} \equiv TDT^{-1} \end{aligned}$$

- Now use the transformation matrix

$$M = 0.5 \begin{bmatrix} 1 & -\mathbf{i} \\ 1 & \mathbf{i} \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & 1 \\ \mathbf{i} & -\mathbf{i} \end{bmatrix}$$

- Then it follows that

$$\begin{aligned} A &= TDT^{-1} = (TM)(M^{-1}DM)(M^{-1}T^{-1}) \\ &= (TM)(M^{-1}DM)(TM)^{-1} \end{aligned}$$

  which has the nice structure:

$$A = \left[\, \mathtt{Re}(e_1) \,\middle|\, \mathtt{Im}(e_1) \,\right] \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \left[\, \mathtt{Re}(e_1) \,\middle|\, \mathtt{Im}(e_1) \,\right]^{-1}$$

  where all the matrices are real.

- With complex roots, diagonalization is to a block diagonal form.

- For this case we have that

$$e^{At} = \left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right] e^{at} \left[\begin{array}{cc} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{array}\right] \left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right]^{-1}$$

- Note that

$$\left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right]^{-1} \left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]$$

- So for an initial condition to excite just this mode, can pick $\mathbf{x}(0) = [\texttt{Re}(e_1)]$, or $\mathbf{x}(0) = [\texttt{Im}(e_1)]$ or a linear combination.

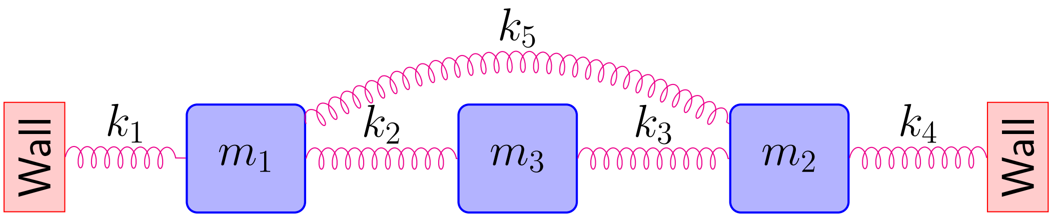
- Example $\mathbf{x}(0) = [\texttt{Re}(e_1)]$

$$\begin{aligned} \mathbf{x}(t) &= e^{At}\mathbf{x}(0) = \left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right] e^{at} \left[\begin{array}{cc} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{array}\right] \cdot \\ &\qquad \left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right]^{-1} [\texttt{Re}(e_1)] \\ &= \left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right] e^{at} \left[\begin{array}{cc} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \end{array}\right] \\ &= e^{at} \left[\begin{array}{c|c} \texttt{Re}(e_1) & \texttt{Im}(e_1) \end{array}\right] \left[\begin{array}{c} \cos(bt) \\ -\sin(bt) \end{array}\right] \\ &= e^{at} \left(\texttt{Re}(e_1)\cos(bt) - \texttt{Im}(e_1)\sin(bt)\right) \end{aligned}$$

which would ensure that only this mode is excited in the response

Example: Spring Mass System

- Classic example: spring mass system consider simple case first: $m_i = 1$, and $k_i = 1$



$$\mathbf{x} = \begin{bmatrix} z_1 & z_2 & z_3 & \dot{z}_1 & \dot{z}_2 & \dot{z}_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix} \quad M = \text{diag}(m_i)$$

$$K = \begin{bmatrix} k_1 + k_2 + k_5 & -k_5 & -k_2 \\ -k_5 & k_3 + k_4 + k_5 & -k_3 \\ -k_2 & -k_3 & k_2 + k_3 \end{bmatrix}$$

- Eigenvalues and eigenvectors of the undamped system

$$\lambda_1 = \pm 0.77\mathbf{i} \quad \lambda_2 = \pm 1.85\mathbf{i} \quad \lambda_3 = \pm 2.00\mathbf{i}$$

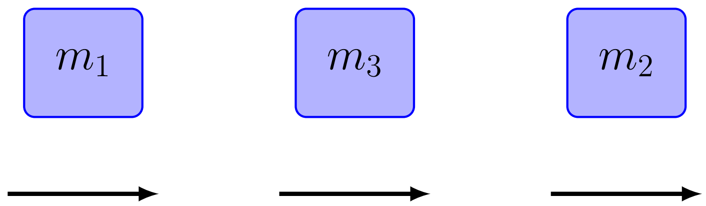
v_1	v_2	v_3
1.00	1.00	1.00
1.00	1.00	-1.00
1.41	-1.41	0.00
$\pm 0.77\mathbf{i}$	$\pm 1.85\mathbf{i}$	$\pm 2.00\mathbf{i}$
$\pm 0.77\mathbf{i}$	$\pm 1.85\mathbf{i}$	$\mp 2.00\mathbf{i}$
$\pm 1.08\mathbf{i}$	$\mp 2.61\mathbf{i}$	0.00

- Initial conditions to excite just the three modes:

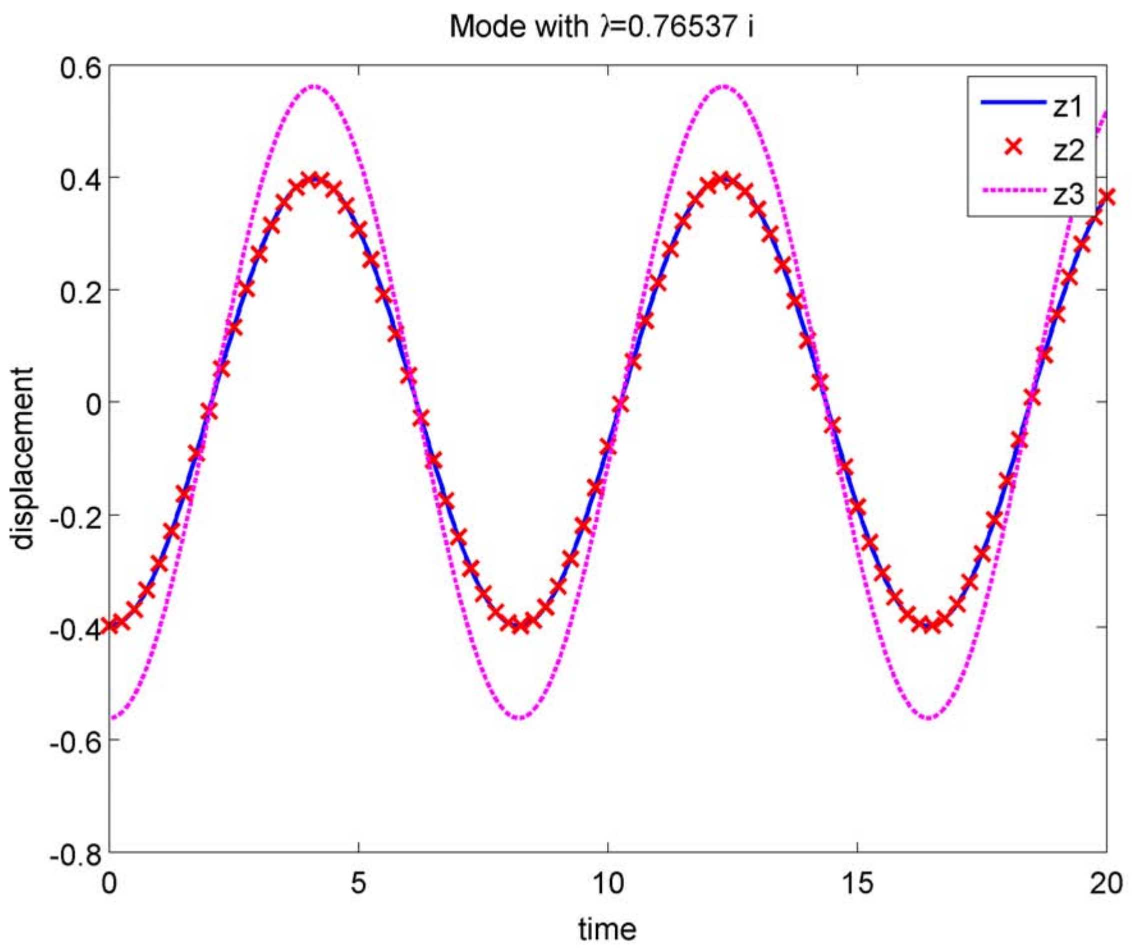
$$\mathbf{x}_i(0) = \alpha_1 \mathrm{Re}(v_i) + \alpha_2 \mathrm{Im}(v_i) \quad \forall \alpha_j \in \mathbb{R}$$

  - Simulation using $\alpha_1 = 1$, $\alpha_2 = 0$

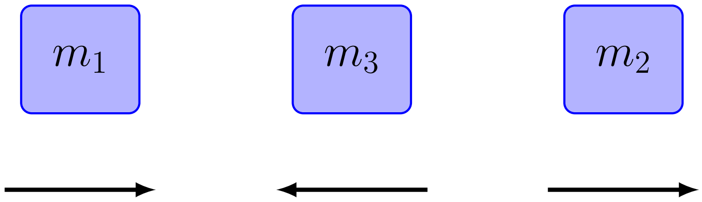
- Visualization important for correct physical interpretation
- Mode 1 $\lambda_1 = \pm 0.77\mathbf{i}$



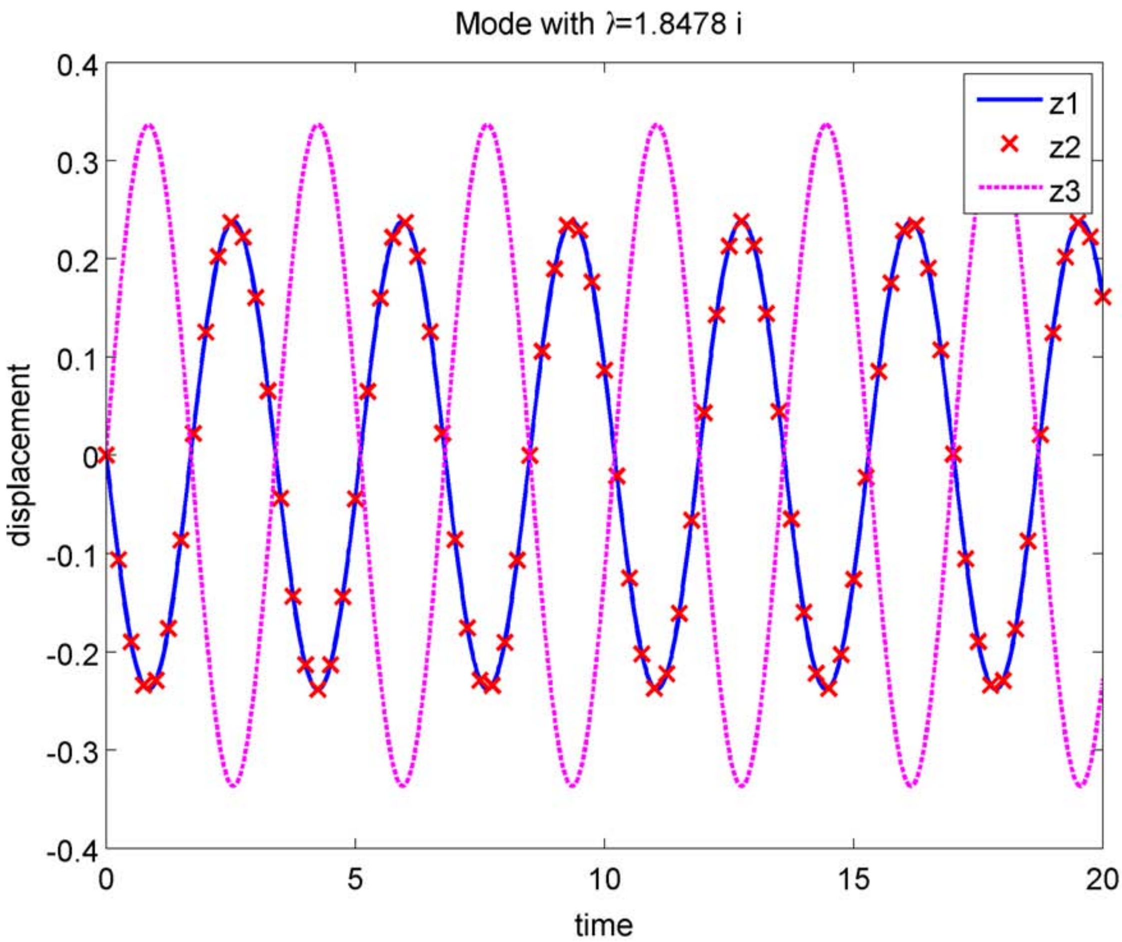
  - Lowest frequency mode, all masses move in same direction
  - Middle mass has higher amplitude motions z_3, motions all in phase



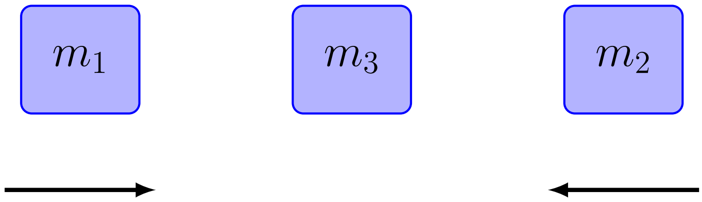
- Mode 2 $\lambda_2 = \pm 1.85\mathbf{i}$



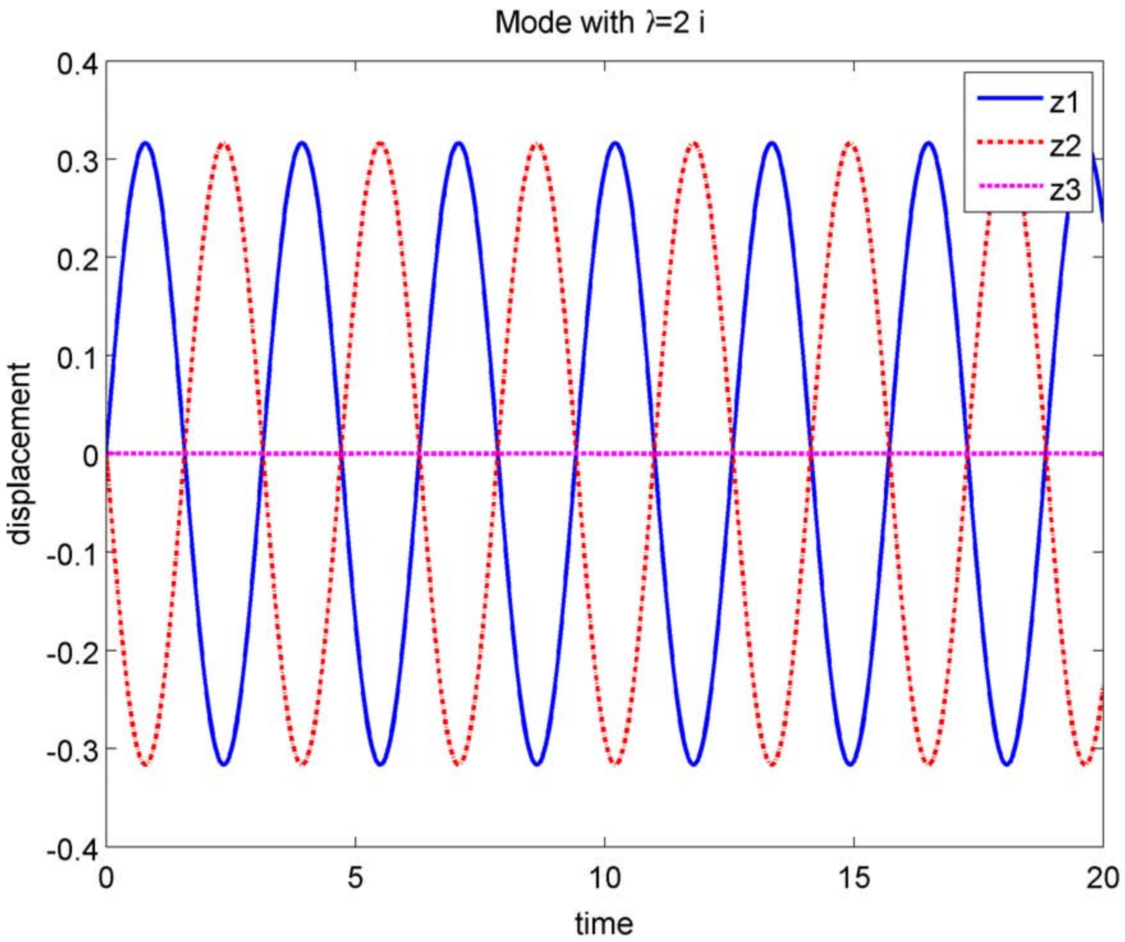
  - Middle frequency mode has middle mass moving in opposition to two end masses
  - Again middle mass has higher amplitude motions z_3



- Mode 3 $\lambda_3 = \pm 2.00\mathbf{i}$

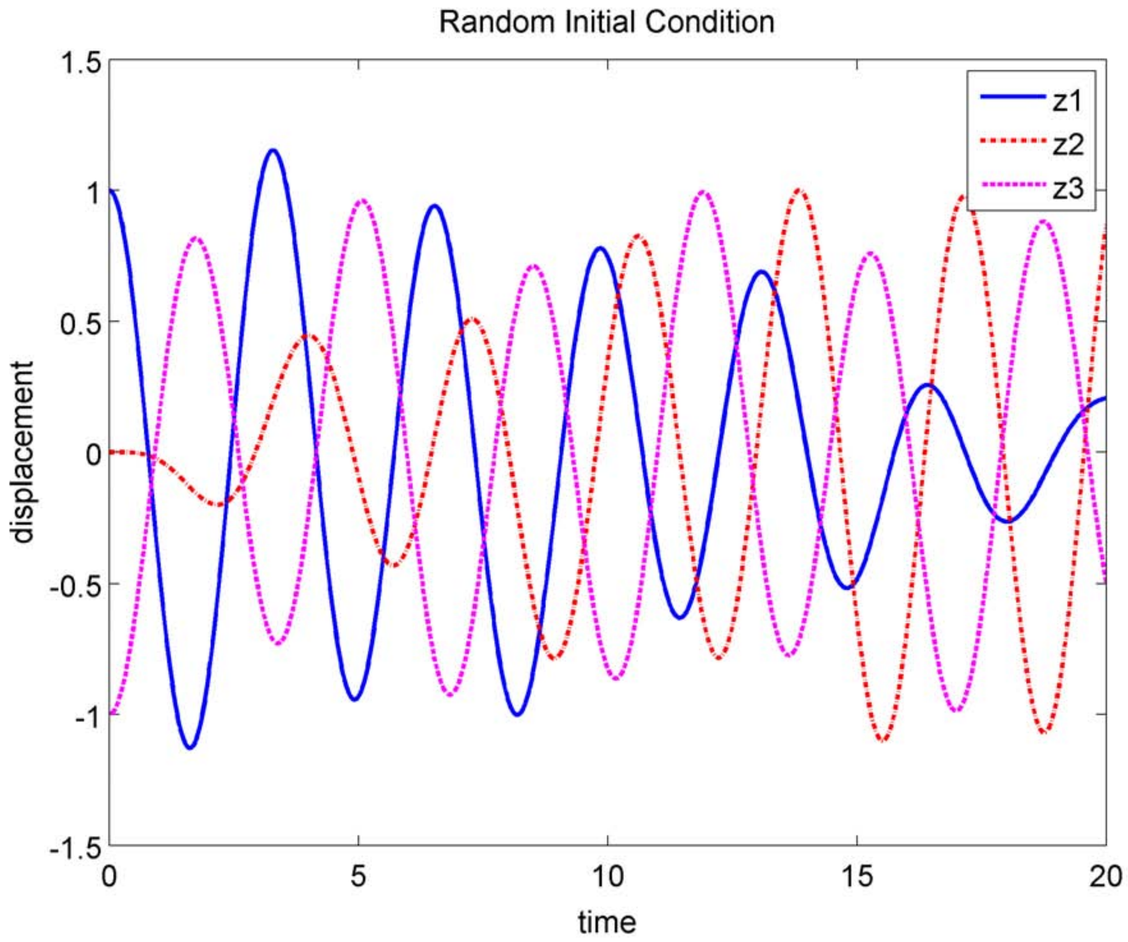


  - Highest frequency mode, has middle mass stationary, and other two masses in opposition



- Eigenvectors that correspond to more constrained motion of the system are associated with higher frequency eigenvalues

- Result if we use a random input is a combination of all three modes



Code: Simulation of Spring Mass System Modes

```
% Simulate modal response for a spring mass system
% Jonathan How, MIT
% Fall 2009
alp1=1; % weighting choice on the IC
m=eye(3); % mass
k=[3 -1 -1;-1 3 -1;-1 -1 2]; % stiffness
a=[m*0 eye(3);-inv(m)*k m*0];
[v,d]=eig(a);
t=[0:.01:20]; l1=1:25:length(t);
G=ss(a,zeros(6,1),zeros(1,6),0);

% use the following to cll the function above
close all
set(0, 'DefaultAxesFontSize', 14, 'DefaultAxesFontWeight','demi')
set(0, 'DefaultTextFontSize', 14, 'DefaultTextFontWeight','demi')
set(0,'DefaultAxesFontName','arial');
set(0,'DefaultTextFontName','arial');set(0,'DefaultlineMarkerSize',10)

figure(1);clf
x0=alp1*real(v(:,1))+(1-alp1)*imag(v(:,1))
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2);hold on
plot(t(l1),x(l1,2),'rx','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2);hold off
xlabel('time');ylabel('displacement')
title(['Mode with \lambda=', num2str(imag(d(1,1))),' i'])
legend('z1','z2','z3')
print -dpng -r300 v1.png

figure(2);clf
x0=alp1*real(v(:,5))+(1-alp1)*imag(v(:,5))
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2);hold on
plot(t(l1),x(l1,2),'rx','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2);hold off
xlabel('time');ylabel('displacement')
title(['Mode with \lambda=', num2str(imag(d(5,5))),' i'])
legend('z1','z2','z3')
print -dpng -r300 v3.png

figure(3);clf
x0=alp1*real(v(:,3))+(1-alp1)*imag(v(:,3))
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2);hold on
plot(t,x(:,2),'r-.','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2);hold off
xlabel('time');ylabel('displacement')
title(['Mode with \lambda=', num2str(imag(d(3,3))),' i'])
legend('z1','z2','z3')
print -dpng -r300 v2.png

figure(4);clf
x0=[1 0 -1 0 0 0]';
[y,t,x]=lsim(G,0*t,t,x0);
plot(t,x(:,1),'-','LineWidth',2)
hold on
plot(t,x(:,2),'r-.','LineWidth',2)
plot(t,x(:,3),'m--','LineWidth',2)
hold off
xlabel('time');ylabel('displacement')
title(['Random Initial Condition'])
legend('z1','z2','z3')
print -dpng -r300 v4.png
```

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