16.30
Estimation and Control of Aerospace Systems
Topic 5 addendum: Signals and Systems

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Outline

Continuous- and discrete-time signals

Continuous-time signal

A (scalar) continuous-time signal is a function that associates to each time $t \in \mathbb{R}$ a real number $y(t)$, i.e., $y : t \mapsto y(t)$. Note: We will use the "standard" (round) parentheses to indicate continuous-time signals.

Discrete-time signal

A (scalar) discrete-time signal is a function that associates to each integer $k \in \mathbb{Z}$ a real number $y[k]$, i.e., $y : k \mapsto y[k]$. Note: We will use the square parentheses to indicate discrete-time signals.
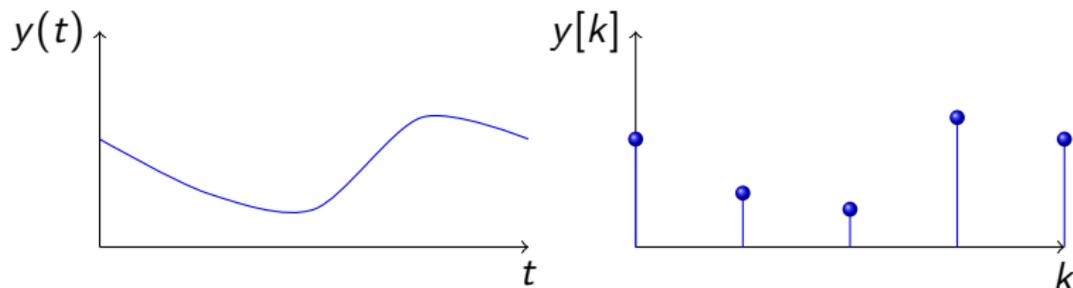
Signals are vectors

Multiplication by a scalar

- Let $\alpha \in \mathbb{R}$. The signal αy can be obtained as:

$$(\alpha y)(t) = \alpha y(t), \quad \text{and} \quad (\alpha y)[k] = \alpha y[k].$$

- Notice $0y$ is always the "zero" signal, where $0(t) = 0$ for all $t \in \mathbb{R}$, and $0[k] = 0$ for all $k \in \mathbb{Z}$, and $1y = y$.

Addition of two signals

- Let u and v be two signals of the same kind (i.e., both in continuous or discrete time).
- The signal $u + v$ is defined as:

$$(u + v)(t) = u(t) + v(t), \quad \text{and} \quad (u + v)[k] = u[k] + v[k].$$

- Notice that $u - u = u + (-1)u = 0$.

Systems

Definition (system)

A system is an operator that transforms an input signal u into a unique output signal y.
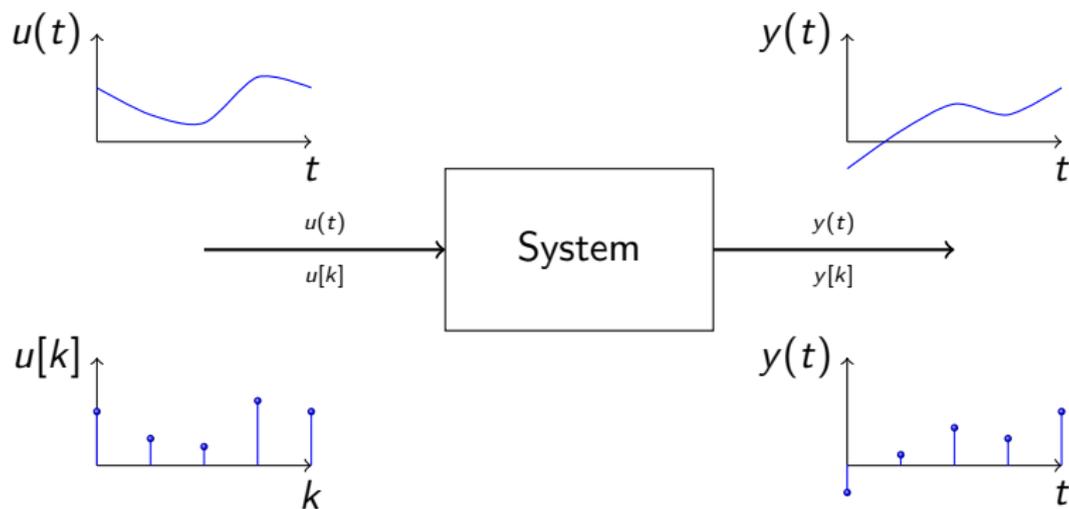
A classification

Continuous-Time System: CT $\to$ CT

This is the kind of systems you studied in 16.06.

Discrete-Time System: DT $\to$ DT

We will study this kind of systems in this class.

Sampler: CT $\to$ DT

This class includes sensors, and A/D (Analog $\to$ Digital) converters.
Let us call a sampler with sampling time T a system such that

$$y[k] = u(kT).$$

Hold: DT $\to$ CT

This class includes actuators, and D/A (Digital $\to$ Analog) converters.
A Zero-Order Hold (ZOH) with holding time T is such that

$$y(t) = u\left[\left\lfloor \frac{t}{T} \right\rfloor\right].$$

Outline

Static/Memoryless systems

Definition (Memoryless system)

A system is said to be memoryless (or static) if, for any $t_0 \in \mathbb{R}$ (resp. $k_0 \in \mathbb{Z}$), the output at time t_0 (resp. at time k_0) depends only on the input at time t_0 (resp. at step k_0).

This is the most basic kind of system. Essentially the output can be written as a "simple" function of the input, e.g.,

$$y(t) = f(u(t)), \qquad y[k] = f(u[k]).$$

Examples:

- A proportional compensator;
- A spring;
- An electrical circuit with resistors only.

In general, systems behave in a more complicated way.

Causality

Definition (Causal system)

A system G is said to be causal if, for any $t_0 \in \mathbb{R}$ (resp., $k_0 \in \mathbb{Z}$) the output at time t_0 (resp. at step k_0) depends only on the input up to, and including, time t_0 (resp. up to, and including, step k_0).

Definition (Strictly causal system)

A system is said to be strictly causal if the dependency is only on the input preceding t_0 (resp., k_0).

A system is causal if it is non-anticipatory, i.e., it cannot respond to inputs that will be applied in the future, but only on past and present inputs. (Strictly causal systems only depend on past inputs).

Note that a static system is causal, but not strictly causal.

Some remarks on causality

Most control systems that are implementable in practice are, in fact, causal. In general, it is not possible to predict the inputs that will be applied in the future.

However, there are some cases in which non-causal systems can actually be interesting to study:

- On-board sensors may provide "look-ahead" information.
  For example, in a nap-of-the-Earth flying mission, the path to the next waypoint, and the altitude profile, may be known in advance.

- Commands may belong to a pre-defined class.
  For example, an autopilot that is programmed to execute an acrobatic maneuver (e.g., a loop).

- Off-line processing.
  For example, a system that is used as a filter to smooth a certain signal, or possibly to "rip" a song from a CD to an MP3 file.

Outline

Time delays

Definition (Time-delay system)

A time-delay system, or more simply a time delay, is a system S_T such that

$$y(t) = S_T(u(t)) = u(t - T) \qquad \text{in continuous time;}$$
$$y[k] = S_T(u[k]) = u[k - T] \qquad \text{in discrete time;}$$
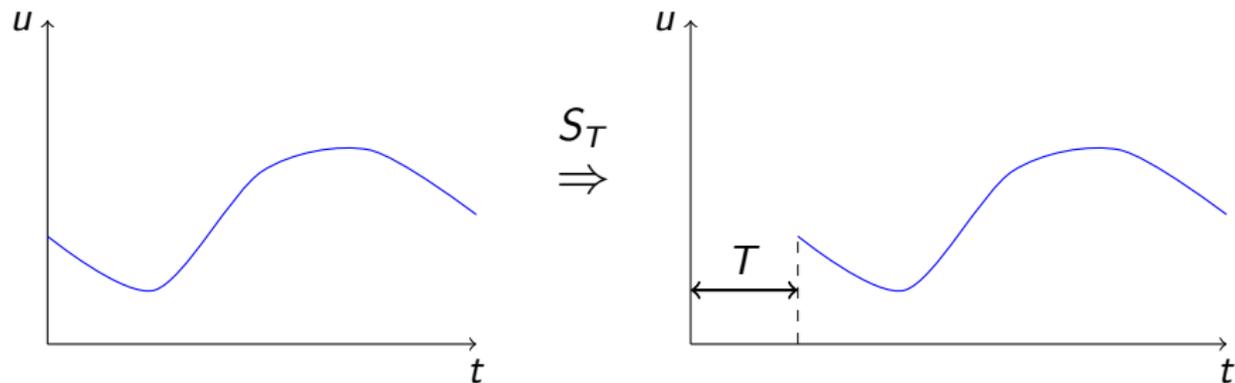
Time invariance

Definition (Time invariant system)

A system is time invariant if it commutes with time delays. In other words, the output to a time-delayed input is the same as the time-delayed output to the original input.
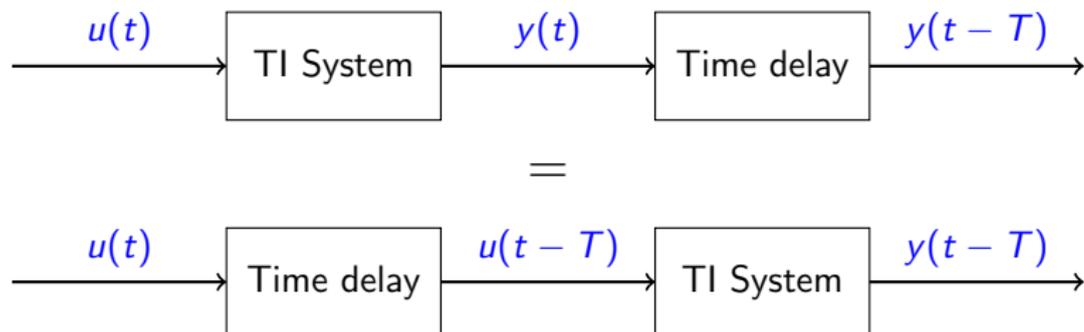
More simply, the response of a time-invariant system does not depend on where you put the origin on the time axis (but it depends on the direction in which time flows).

Outline

State of a system

We know that, if a system is causal, in order to compute its output at a given time t_0, we need to know "only" the input signal over $(-\infty, t_0]$. (Similarly for DT systems.)

This is a lot of information. Can we summarize it with something more manageable?

Definition (state)

The state $x(t_1)$ of a causal system at time t_1 (resp. at step k_1) is the information needed, together with the input u between times t_1 and t_2 (resp. between steps k_1 and k_2), to uniquely predict the output at time t_2 (resp. at step k_2), for all $t_2 \geq t_1$ (resp. for all steps $k_2 \geq k_1$).

In other words, the state of the system at a given time summarizes the whole history of the past inputs $-\infty$, for the purpose of predicting the output at future times.

Usually, the state of a system is a vector in some n-dimensional space $\mathbb{R}^n$.

Dimension of a system

The choice of a state for a system is not unique (in fact, there are infinite choices, or realizations).

However, there are come choices of state which are preferable to others; in particular, we can look at "minimal" realizations.

Definition (Dimension of a system)

We define the dimension of a causal system as the minimal number of variables sufficient to describe the system's state (i.e., the dimension of the smallest state vector).

We will deal mostly with finite-dimensional systems, i.e., systems which can be described with a finite number of variables.

Some remarks on infinite-dimensional systems

Even though we will not address infinite-dimensional systems in detail, some examples are very useful:

- (CT) Time-delay systems: Consider the very simple time delay S_T, defined as a continuous-time system such that its input and outputs are related by

$$y(t) = u(t - T).$$

  In order to predict the output at times after t, the knowledge of the input for times in $(t - T, t]$ is necessary.

- PDE-driven systems: Many systems in aerospace, arising, e.g., in structural control and flow control applications, can only be described exactly using a continuum of state variables (stress, displacement, pressure, temperature, etc.). These are infinite-dimensional systems.

In order to deal with infinite-dimensional systems, approximate discrete models are often used to reduce the dimension of the state.

Outline

Linear Systems

Definition (Linear system)

A system is said a Linear System if, for any two inputs u_1 and u_2, and any two real numbers α, β, the following are satisfied:

$$
\begin{aligned}
u_1 &\rightarrow y_1, \\
u_2 &\rightarrow y_2, \\
\alpha u_1 + \beta u_2 &\rightarrow \alpha y_1 + \beta y_2.
\end{aligned}
$$

Superposition of effects

This property is very important: it tells us that if we can decompose a complicated input into a sum of simple signals, we can obtain the output as the sum of the individual outputs corresponding to the simple inputs. Examples (in CT, same holds in DT):

- Taylor series: $u(t) = \sum_{i=0}^{\infty} c_i t^i$.
- Fourier series: $u(t) = \sum_{i=0}^{\infty} \left(a_i \cos(i\,t) + b_i \sin(i\,t) \right)$.

State-space model

Finite-dimensional linear systems can always be modeled using a set of differential (or difference) equations as follows:

Definition (Continuous-time systems)

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t);$$
$$y(t) = C(t)x(t) + D(t)u(t);$$

Definition (Discrete-time systems)

$$x[k+1] = A[k]x[k] + B[k]u[k];$$
$$y[k] = C[k]x[k] + D[k]u[k];$$

The matrices appearing in the above formulas are in general functions of time, and have the correct dimensions to make the equations meaningful.

LTI State-space model

If the system is Linear Time-Invariant (LTI), the equations simplify to:

Definition (Continuous-time systems)

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t);$$
$$y(t) = Cx(t) + Du(t);$$

Definition (Discrete-time systems)

$$x[k+1] = Ax[k] + Bu[k];$$
$$y[k] = Cx[k] + Du[k];$$

In the above formulas, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, $D \in \mathbb{R}$, and n is the dimension of the state vector.

Example of DT system: accumulator

- Consider a system such that

$$y[k] = \sum_{i=-\infty}^{k-1} u[i].$$

- Notice that we can rewrite the above as

$$y[k] = \left(\sum_{i=-\infty}^{k-2} u[i] \right) + u[k-1] = y[k-1] + u[k-1].$$

- In other words, we can set $x[k] = y[k]$ as a state, and get the following state-space model:

$$\begin{array}{rcl} x[k+1] & = & x[k] + u[k], \\ y[k] & = & x[k]. \end{array}$$

- Let $x[0] = y[0] = 0$, and $u[k] = 1$; we can solve by repeated substitution:

$$\begin{array}{rcl} x[1] & = & x[0] + u[0] = 0 + 1 = 1, \quad y[1] = x[1] = 1; \\ x[2] & = & x[1] + u[1] = 1 + 1 = 2, \quad y[2] = x[2] = 2; \\ & \cdots & \\ x[k] & = & x[k-1] + u[k-1] = k-1+1 = k, \quad y[k] = x[k] = k; \end{array}$$

LTI State-space models in Matlab

```
>> A= [1, 0.1; 0, 1]; B= [1; 2]; C = [3, 4]; D=0;
>> P = ss(A,B,C,D)

a =
        x1   x2
   x1    1   0.1
   x2    0   1

b =
        u1
   x1    1
   x2    2

c =
        x1  x2
   y1    3   4

d =
        u1
   y1    0

Continuous-time model.
```

LTI State-space models in Matlab

Example (Discrete-time LTI system)

```
>> A= [1, 0.1; 0, 1]; B= [1; 2]; C = [3, 4]; D=0; Ts = 0.2;
>> P = ss(A,B,C,D,Ts)

a =
        x1   x2
   x1   1   0.1
   x2   0    1

b =
        u1
   x1   1
   x2   2

c =
        x1  x2
   y1   3   4

d =
        u1
   y1   0

Sampling time: 0.2
Discrete-time model.
```

Finite-dimensional Linear Systems 1/2

- Recall the definition of a linear system. Essentially, a system is linear if the linear combination of two inputs generates an output that is the linear combination of the outputs generated by the two individual inputs.

- The definition of a state allows us to summarize the past inputs into the state, i.e.,

$$u(t), -\infty \leq t \leq +\infty \qquad \Leftrightarrow \qquad \left\{ \begin{array}{l} x(t_0), \\ u(t), \quad t \geq t_0, \end{array} \right.$$

  (similar formulas hold for the DT case.)

- We can extend the definition of linear systems as well to this new notion.

Finite-dimensional Linear Systems 2/2

Definition (Linear system (again))

A system is said a Linear System if, for any $u_1, u_2, t_0, x_{0,1}, x_{0,2}$, and any two real numbers α, β, the following are satisfied:

$$\begin{cases} x(t_0) = x_{0,1}, \\ u(t) = u_1(t), \quad t \geq t_0, \end{cases} \rightarrow y_1,$$

$$\begin{cases} x(t_0) = x_{0,2}, \\ u(t) = u_2(t), \quad t \geq t_0, \end{cases} \rightarrow y_2,$$

$$\begin{cases} x(t_0) = \alpha x_{0,1} + \beta x_{0,2}, \\ u(t) = \alpha u_1(t) + \beta u_2(t), \quad t \geq t_0, \end{cases} \rightarrow \alpha y_1 + \beta y_2.$$

Similar formulas hold for the discrete-time case.

Forced response and initial-conditions response

- Assume we want to study the output of a system starting at time t_0, knowing the initial state $x(t_0) = x_0$, and the present and future input $u(t)$, $t \geq t_0$. Let us study the following two cases instead:

  - Initial-conditions response:

  $$\left\{ \begin{array}{l} x_{\mathrm{IC}}(t_0) = x_0, \\ u_{\mathrm{IC}}(t) = 0, \quad t \geq t_0, \end{array} \right. \quad \rightarrow \quad y_{\mathrm{IC}};$$

  - Forced response:

  $$\left\{ \begin{array}{l} x_{\mathrm{F}}(t_0) = 0, \\ u_{\mathrm{F}}(t) = u(t), \quad t \geq t_0, \end{array} \right. \quad \rightarrow \quad y_{\mathrm{F}}.$$

- Clearly, $x_0 = x_{\mathrm{IC}} + x_{\mathrm{F}}$, and $u = u_{\mathrm{IC}} + u_{\mathrm{F}}$, hence

  $$y = y_{\mathrm{IC}} + y_{\mathrm{F}},$$

  that is, we can always compute the output of a linear system by adding the output corresponding to zero input and the original initial conditions, and the output corresponding to a zero initial condition, and the original input.

- In other words, we can study separately the effects of non-zero inputs and of non-zero initial conditions. The "complete" case can be recovered from these two.

16.30 / 16.31 Feedback Control Systems
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