Topic #3

16.30/31 Feedback Control Systems

Frequency response methods

- Analysis
- Synthesis
- Performance
- Stability in the Frequency Domain
- Nyquist Stability Theorem

FR: Introduction

- Root locus methods have:
  - Advantages:
    - Good indicator of transient response;
    - Explicitly shows location of all closed-loop poles;
    - Trade-offs in the design are fairly clear.
  - Disadvantages:
    - Requires a transfer function model (poles and zeros);
    - Difficult to infer all performance metrics;
    - Hard to determine response to steady-state (sinusoids)
    - Hard to infer stability margins

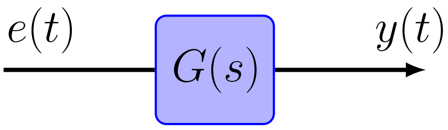
- Frequency response methods are a good complement to the root locus techniques:
  - Can infer performance and stability from the same plot
  - Can use measured data rather than a transfer function model
  - Design process can be independent of the system order
  - Time delays are handled correctly
  - Graphical techniques (analysis and synthesis) are quite simple.

Frequency Response Function

- Given a system with a transfer function $G(s)$, we call the $G(\mathbf{j}\omega)$, $\omega \in [0, \infty)$ the **frequency response function** (FRF)

$$G(\mathbf{j}\omega) = |G(\mathbf{j}\omega)| \measuredangle G(\mathbf{j}\omega)$$

  - The FRF can be used to find the **steady-state** response of a system to a sinusoidal input since, if



and $e(t) = \sin 2t$, $|G(2\mathbf{j})| = 0.3$, $\measuredangle G(2\mathbf{j}) = -80^\circ$, then the steady-state output is

$$y(t) = 0.3 \sin(2t - 80^\circ)$$

$\Rightarrow$ The FRF clearly shows the magnitude (and phase) of the response of a system to sinusoidal input

- A variety of ways to display this:

1. Polar (**Nyquist**) plot – Re vs. Im of $G(\mathbf{j}\omega)$ in complex plane.
   - Hard to visualize, not useful for synthesis, but gives definitive tests for stability and is the basis of the robustness analysis.
2. **Nichols** Plot – $|G(\mathbf{j}\omega)|$ vs. $\measuredangle G(\mathbf{j}\omega)$, which is very handy for systems with lightly damped poles.
3. **Bode** Plot – Log $|G(\mathbf{j}\omega)|$ and $\measuredangle G(\mathbf{j}\omega)$ vs. Log frequency.
   - Simplest tool for visualization and synthesis
   - Typically plot 20$\log |G|$ which is given the symbol dB

- Use logarithmic since if

$$\begin{aligned}\log|G(s)| &= \left|\frac{(s+1)(s+2)}{(s+3)(s+4)}\right| \\ &= \log|s+1| + \log|s+2| - \log|s+3| - \log|s+4|\end{aligned}$$

  and each of these factors can be calculated separately and then added to get the total FRF.

- Can also split the phase plot since

$$\measuredangle\frac{(s+1)(s+2)}{(s+3)(s+4)} = \begin{aligned}&\measuredangle(s+1) + \measuredangle(s+2) \\ &-\measuredangle(s+3) - \measuredangle(s+4)\end{aligned}$$

- The keypoint in the sketching of the plots is that good straightline approximations exist and can be used to obtain a good prediction of the system response.

Bode Example

- Draw Bode for

$$G(s) = \frac{s+1}{s/10+1}$$

$$|G(\mathbf{j}\omega)| = \frac{|\mathbf{j}\omega+1|}{|\mathbf{j}\omega/10+1|}$$

$$\log|G(\mathbf{j}\omega)| = \log[1+(\omega/1)^2]^{1/2} - \log[1+(\omega/10)^2]^{1/2}$$

- Approximation

$$\log[1+(\omega/\omega_i)^2]^{1/2} \approx \begin{cases} 0 & \omega \ll \omega_i \\ \log[\omega/\omega_i] & \omega \gg \omega_i \end{cases}$$

Two straightline approximations that intersect at $\omega \equiv \omega_i$

- Error at ω_i obvious, but not huge and the straightline approximations are very easy to work with.

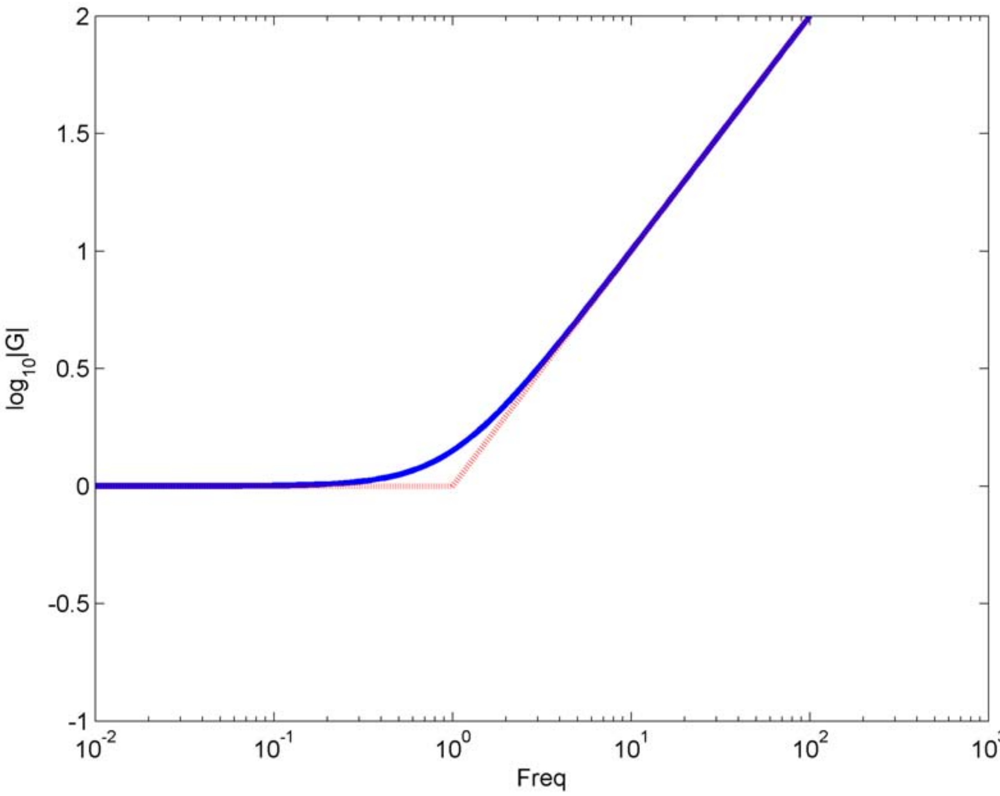


Fig. 1: Frequency response basic approximation

- To form the composite sketch,
  - Arrange representation of transfer function so that DC gain of each element is unity (except for parts that have poles or zeros at the origin) – absorb the gain into the overall plant gain.
  - Draw all component sketches
  - Start at low frequency (DC) with the component that has the lowest frequency pole or zero (i.e. s=0)
  - Use this component to draw the sketch up to the frequency of the next pole/zero.
  - Change the slope of the sketch at this point to account for the new dynamics: -1 for pole, +1 for zero, -2 for double poles, ...
  - Scale by overall DC gain

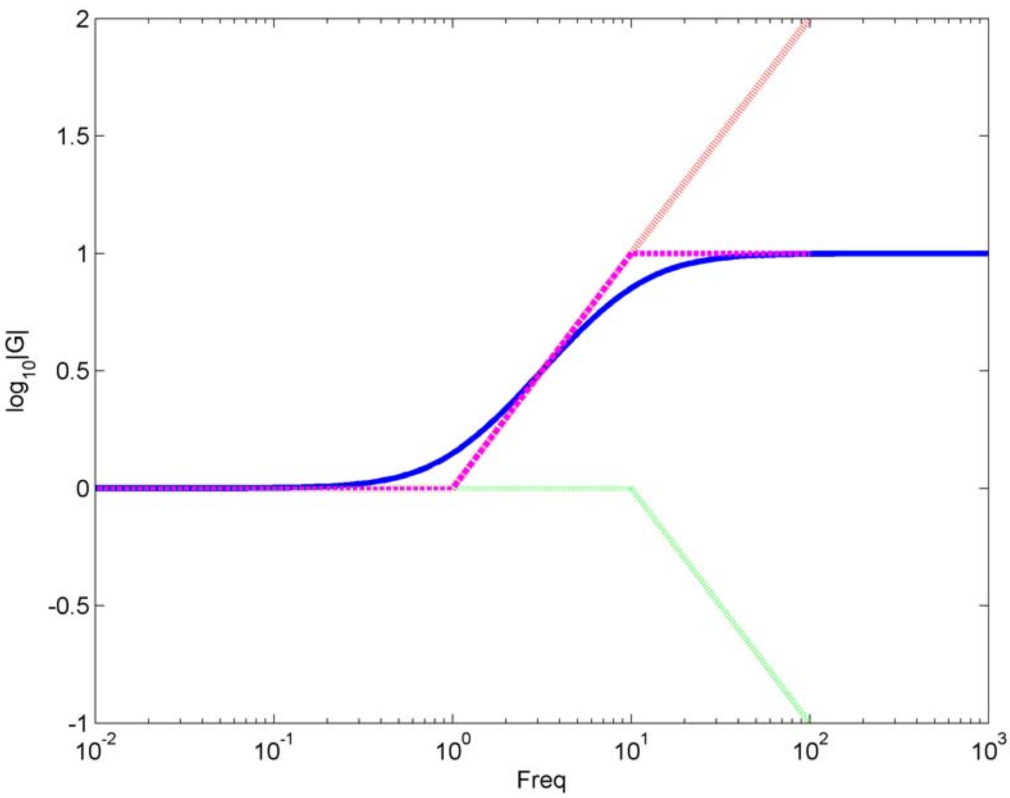


Fig. 2: $G(s) = 10(s+1)/(s+10)$ which is a lead.

- Since $\measuredangle G(\mathbf{j}\omega) = \measuredangle(1+\mathbf{j}\omega) - \measuredangle(1+\mathbf{j}\omega/10)$, we can construct phase plot for complete system in a similar fashion
  - Know that $\measuredangle(1+\mathbf{j}\omega/\omega_i) = \tan^{-1}(\omega/\omega_i)$
- Can use straightline approximations

$$\measuredangle(1+\mathbf{j}\omega/\omega_i) \approx \begin{cases} 0 & \omega/\omega_i \leq 0.1 \\ 90^\circ & \omega/\omega_i \geq 10 \\ 45^\circ & \omega/\omega_i = 1 \end{cases}$$

- Draw components using breakpoints that are at $\omega_i/10$ and $10\omega_i$

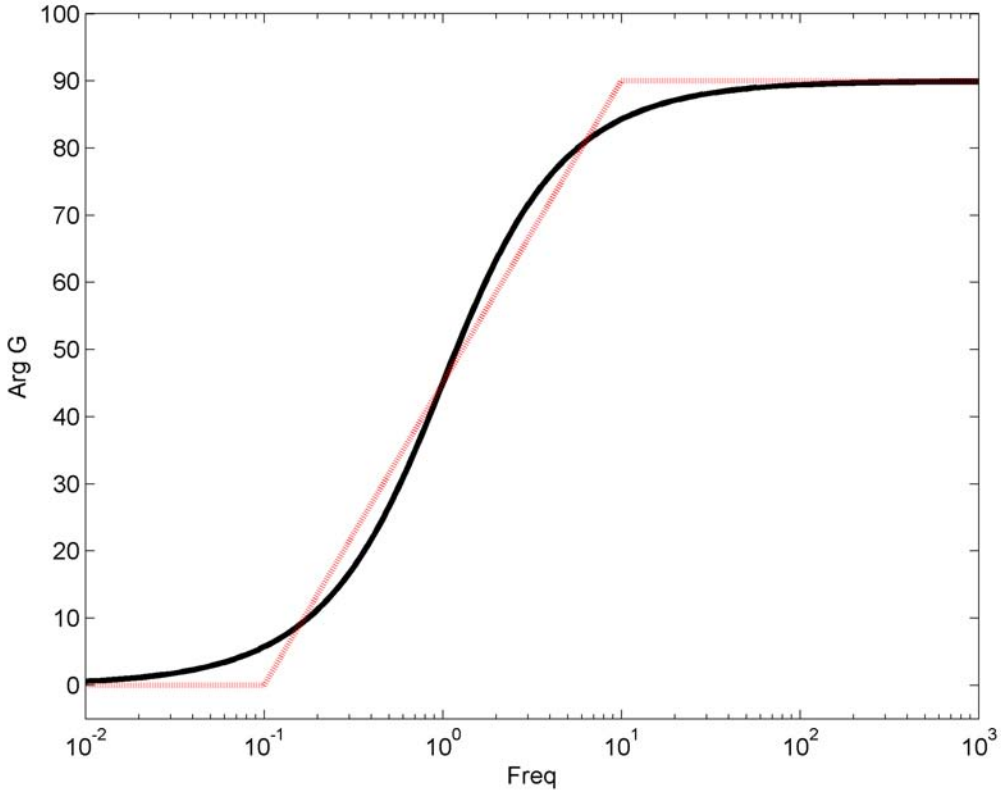


Fig. 3: Phase plot for $(s+1)$

- Then add them up starting from zero frequency and changing the slope as $\omega \to \infty$

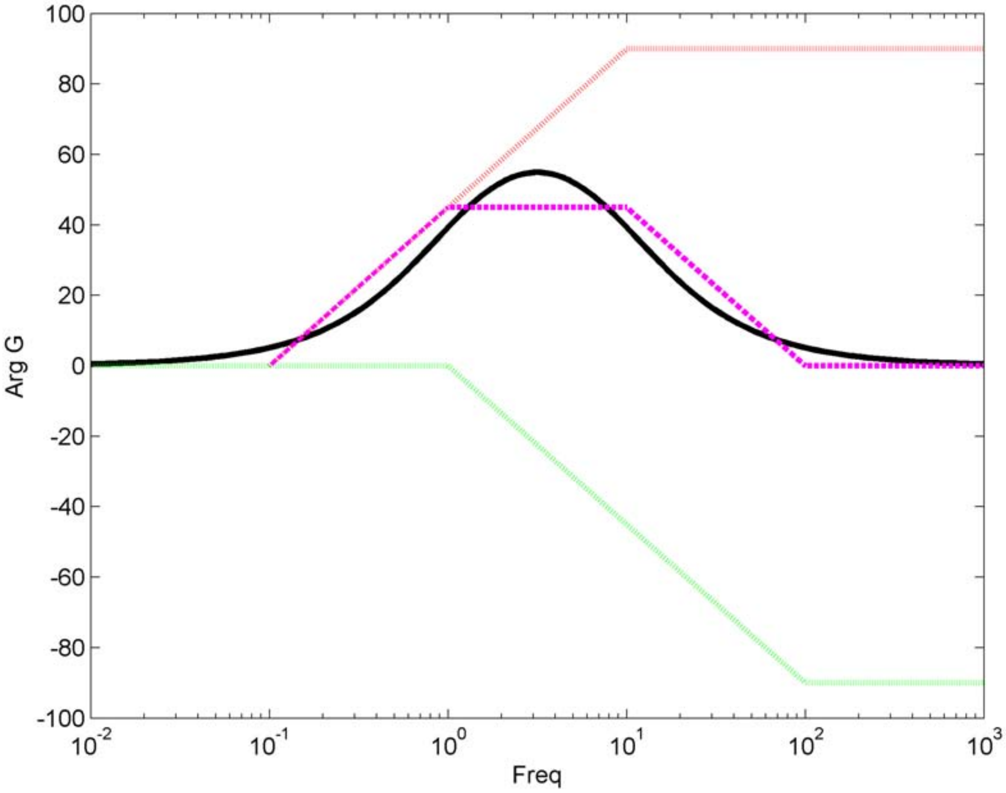


Fig. 4: Phase plot $G(s) = 10(s + 1)/(s + 10)$ which is a “lead”.

Frequency Stability Tests

- Want tests on the loop transfer function $L(s) = G_c(s)G(s)$ that can be performed to establish stability of the closed-loop system

$$G_{cl}(s) = \frac{G_c(s)G(s)}{1 + G_c(s)G(s)}$$

  - Easy to determine using a root locus.
  - How do this in the frequency domain? i.e., what is the simple equivalent of the statement "does root locus go into RHP"?

- **Intuition**: All points on the root locus have the properties that

$$\measuredangle L(s) = \pm 180^\circ \text{ and } |L(s)| = 1$$

  - So at the point of neutral stability (i.e., imaginary axis crossing), we know that these conditions must hold for $s = \mathbf{j}\omega$
  - So for neutral stability in the Bode plot (assume stable plant), must have that $\measuredangle L(\mathbf{j}\omega) = \pm 180^\circ$ and $|L(\mathbf{j}\omega)| = 1$
  - So for most systems we would expect to see $|L(\mathbf{j}\omega)| < 1$ at the frequencies ω_π for which $\measuredangle L(\mathbf{j}\omega_\pi) = \pm 180^\circ$

- Note that $\measuredangle L(\mathbf{j}\omega) = \pm 180^\circ$ and $|L(\mathbf{j}\omega)| = 1$ corresponds to $L(\mathbf{j}\omega) = -1 + 0\mathbf{j}$

Gain and Phase Margins

- **Gain Margin:** factor by which the gain is less than 1 at the frequencies ω_π for which $\measuredangle L(\mathbf{j}\omega_\pi) = 180^\circ$

$$GM = -20 \log |L(\mathbf{j}\omega_\pi)|$$

- **Phase Margin:** angle by which the system phase differs from 180° when the loop gain is 1.
  - Let ω_c be the frequency at which $|L(\mathbf{j}\omega_c)| = 1$, and $\phi = \measuredangle L(\mathbf{j}\omega_c)$ (typically less than zero), then

$$PM = 180^\circ + \phi$$

- Typical stable system needs both $GM > 0$ and $PM > 0$

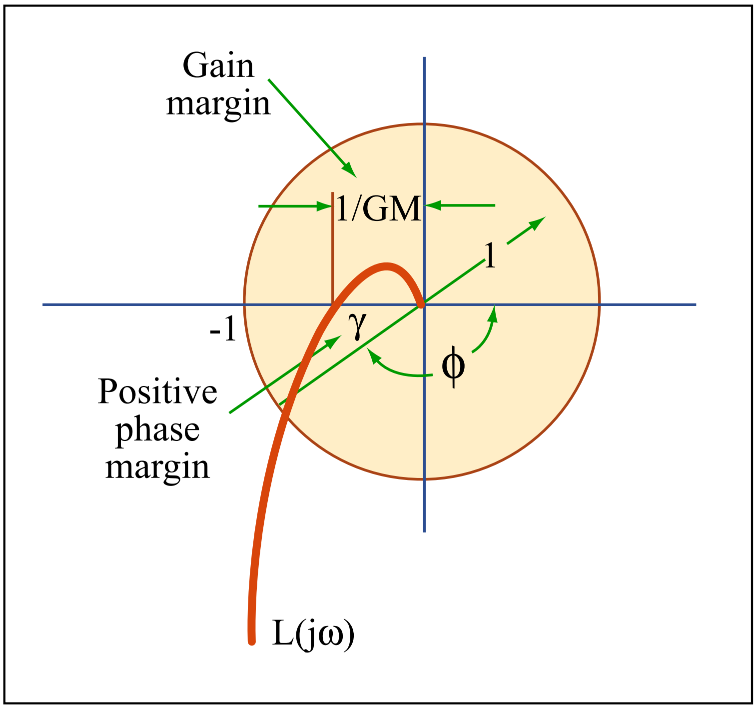


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Fig. 5: Gain and Phase Margin for stable system in a polar plot

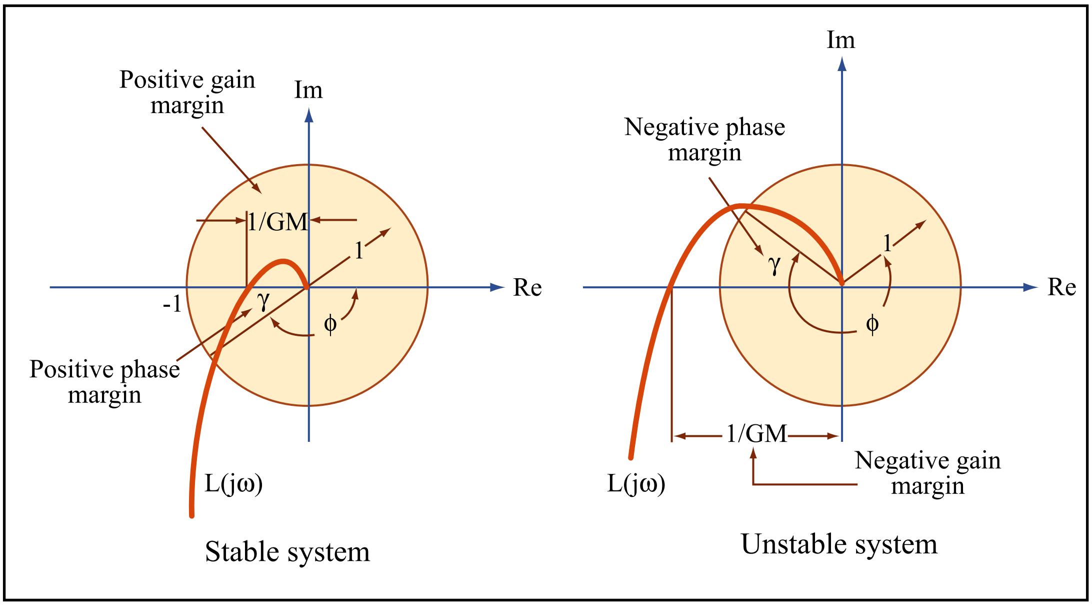

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Fig. 6: Gain and Phase Margin in Polar plots

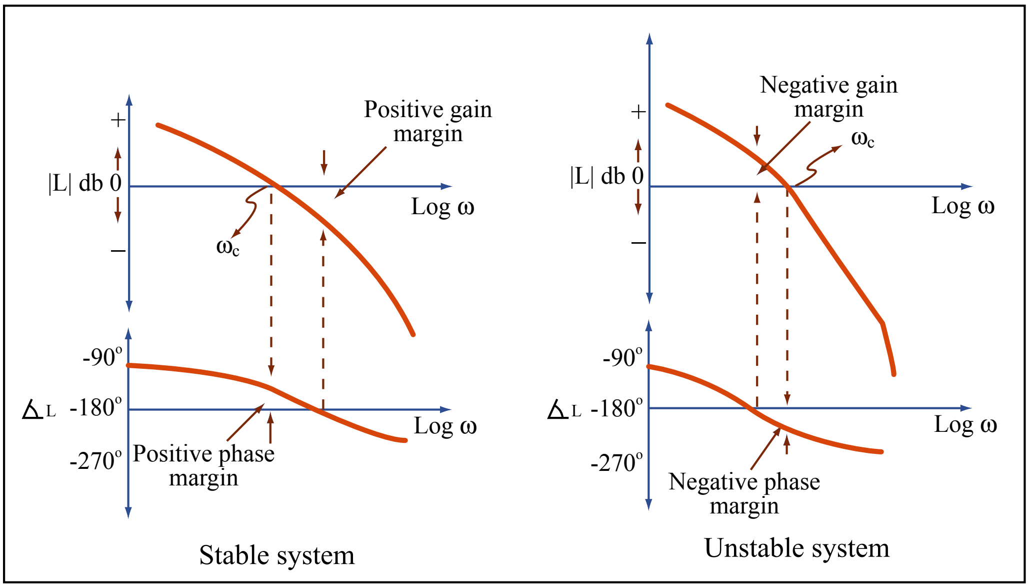

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Fig. 7: Gain and Phase Margin in Bode plots

- Can often predict closed-loop stability looking at the GM and PM

- So the test for **neutral stability** is whether, at some frequency, the plot of $L(\mathbf{j}\omega)$ in the complex plane passes through the **critical point** $s = -1$

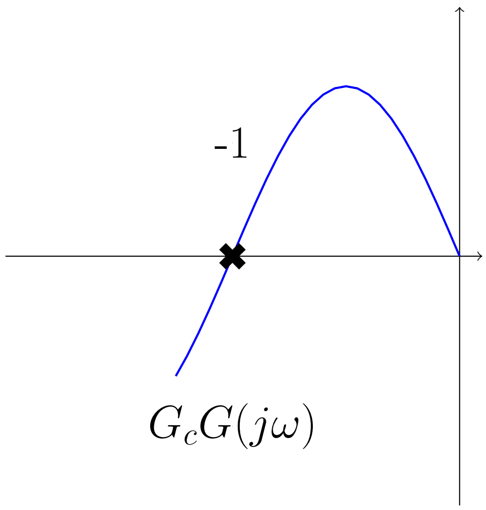


Fig. 8: Polar plot of a neutrally stable case

- This is good intuition, but we need to be careful because the previous statements are only valid if we assume that:
  - Increasing gain leads to instability
  - $|L(\mathbf{j}\omega)| = 1$ at only 1 frequency

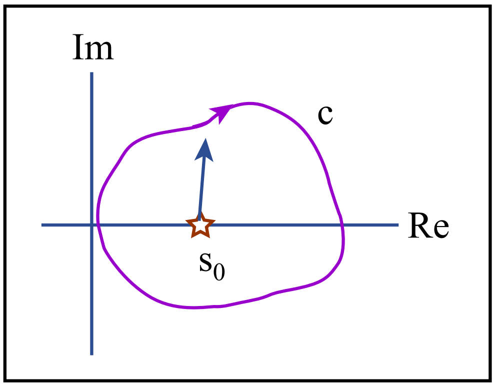
  which are reasonable assumptions, but **not** always valid.

- In particular, if $L(s)$ is unstable, this prediction is a little more complicated, and it can be hard to do in a Bode diagram $\Rightarrow$ need more precise test.

- A more precise version must not only consider whether $L(s)$ passes through -1, but how many times it **encircles it**.
  - In the process, we must take into account the stability of $L(s)$

Nyquist Stability

- **Key pieces**: an **encirclement** – an accumulation of of 360° of phase by a vector (tail at s_0) as the tip traverses the contour $c \Rightarrow c$ encircles s_0



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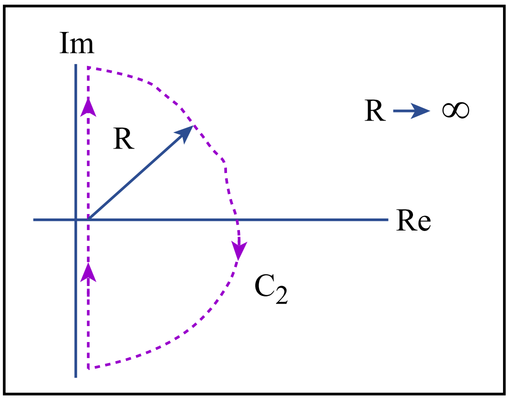
- We are interested in the plot of $L(s)$ for a very specific set of values of s, called the **Nyquist Path**.



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  - Case shown assumes that $L(s)$ has no imaginary axis poles, which is where much of the complexity of plotting occurs.
  - Also note that if $\lim_{s\to\infty} L(s) = 0$, then much of the plot of $L(s)$ for values of s on the Nyquist Path is at the origin.

- **Nyquist Diagram**: plot of $L(s)$ as s moves around the Nyquist path C_2

- Steps:
  - Construct Nyquist Path for particular $L(s)$
  - Draw Nyquist Diagram
  - Count # of encirclements of the critical point -1
- Why do we care about the # of encirclements?
  - Turns out that (see appendix) that if $L(s)$ has any poles in the RHP, then the Nyquist diagram/plot **must** encircle the critical point -1 for the closed-loop system to be stable.
- It is our job to ensure that we have enough encirclements – how many do we need?

- **Nyquist Stability Theorem:**
  - P = # poles of $L(s) = G(s)G_c(s)$ in the RHP
  - Z = # closed-loop poles in the RHP
  - N = # clockwise encirclements of the Nyquist Diagram about the *critical point* -1.

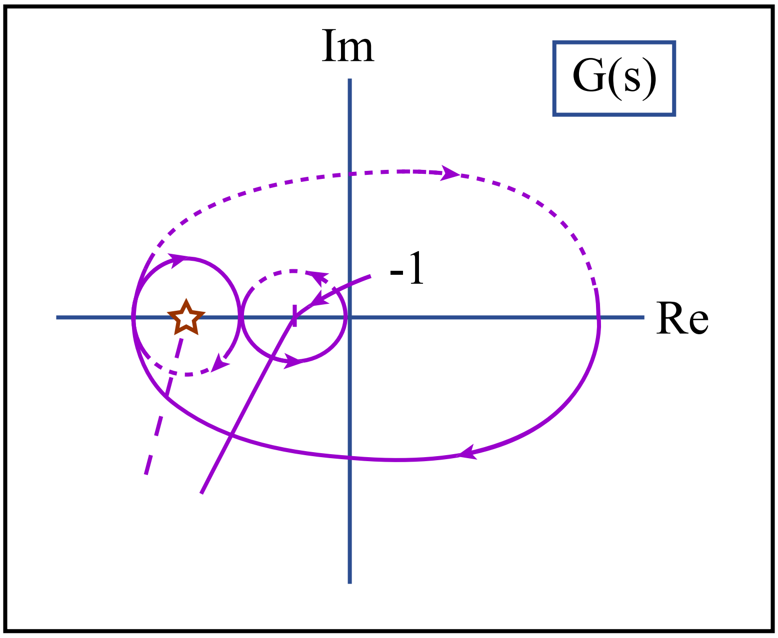
  Can show that $Z = N + P$
  $\Rightarrow$ So for the closed-loop system to be stable (i.e., no closed-loop poles in the RHP), need

$$Z \triangleq 0 \quad \Rightarrow \quad N = -P$$

  - Note that since $P \geq 0$, then would expect CCW encirclements

- The whole issue with the Nyquist test boils down to developing a robust way to make accurate plots and count N.
  - Good approach to find the # of crossing from a point s_0 is:
    - Draw a line from s_0
    - Count # of times that line and the Nyquist plot cross

$$N = \#CW_{\text{crossings}} - \#CCW_{\text{crossings}}$$



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- **Observation**: If the stability of the system is unclear from the Bode diagram, then always revert to the Nyquist plot.

FR: Summary

- Bode diagrams are easy to draw
- Will see that control design is relatively straight forward as well
- Can be a bit complicated to determine stability, but this is a relatively minor problem and it is easily handled using Nyquist plots
- Usually only necessary to do one of Bode/Root Locus analysis, but they do provide different perspectives, so I tend to look at both in `sisotool`.

- Nyquist test gives us the desired frequency domain stability test
  - Corresponds to a test on the number of encirclements of the critical point
- For most systems that can be interpreted as needing the $GM > 0$ and $PM > 0$
  - Typically design to $GM \sim 6dB$ and $PM \sim 30^\circ - 60^\circ$
- Introduced $S(s)$ as a basic measure of system robustness.

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